

# Perturbative Renormalizability of Chiral Two Pion Exchange in Nucleon-Nucleon Scattering: P- and D-waves

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We study the perturbative renormalizability of chiral two pion exchange in nucleon-nucleon scattering for p- and d-waves within the effective field theory approach. The one pion exchange potential is fully iterated at the leading order in the expansion, a choice generating a consistent and well-defined power counting that we explore in detail. The results show that perturbative chiral two pion exchange reproduces the data up to a center-of-mass momentum of  $k_{\text{cm}} \sim 300$  MeV at next-to-next-to-leading order and that the effective field theory expansion convergences up to  $k_{\text{cm}} \sim 350$  MeV.

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## I. INTRODUCTION

The nature and derivation of the nucleon-nucleon interaction is probably the central problem of nuclear physics. The modern point of view is that any serious theoretical formulation of the nuclear force should be grounded on quantum chromodynamics (QCD), the fundamental theory of the strong interactions. However direct calculations on the lattice are still unavailable at the physical pion mass, despite promising results which are beginning to appear in the two [1, 2] and three [3] nucleon sectors at larger pion masses. A different, more indirect path is provided by the effective field theory (EFT) formulation of the nuclear forces [4–8], in which the pion exchanges are constrained by the requirement of broken chiral symmetry, the main low energy manifestation of QCD. This approach exploits the well-known separation of scales appearing in the two-nucleon system with the purpose of generating a systematic and model independent low energy expansion of the nuclear potential and the scattering amplitudes. In addition, nuclear EFT may potentially bridge current lattice QCD calculations with physical results by means of chiral extrapolations.

In the nuclear EFT proposed by Weinberg [9, 10] the nucleon-nucleon potential is expanded as a power series (i.e. a power counting) in terms of the parameter  $Q/\Lambda_0$

$$V_{\text{NN}}(r) = V^{(0)}(r) + V^{(2)}(r) + V^{(3)}(r) + \mathcal{O}\left(\frac{Q^4}{\Lambda_0^4}\right), \quad (1)$$

where  $Q$  represents the low-energy scales of the system, usually the momentum  $p$  exchanged between the nucleons and the pion mass  $m_\pi$ , while  $\Lambda_0$  stands for the high-energy scales that are not explicitly taken into account in

the theory, e.g. the rho meson mass  $m_\rho$ . The full potential is then iterated into the Schrödinger or Lippmann-Schwinger equation [11–33], or more recently within the nuclear lattice approach [34–42]<sup>1</sup>, in order to obtain theoretical predictions. Regarding the notation, order  $Q^0$  will also be referred to as leading order (LO), order  $Q^2$  as next-to-leading order (NLO), order  $Q^3$  as next-to-next-to-leading order (N<sup>2</sup>LO), and so on.

Among the advantages of the Weinberg prescription we can count that it observes the non-perturbative nature of the nuclear forces and naturally fits into the traditional nuclear physics paradigm of employing a potential as the basic calculational input. In particular, the construction of potential descriptions of the nuclear force is conceptually straightforward, as the procedure only involves the expansion of the chiral potential up to a given order and the determination of the free parameters of the theory by a fit to the available low-energy scattering data. Eventually, at high enough order, in particular N<sup>3</sup>LO, this procedure leads to chiral potentials that are able to reproduce the two-nucleon scattering data below a laboratory energy of  $E_{\text{lab}} = 300$  MeV with a  $\chi^2/d.o.f. \simeq 1$  [17, 20], a remarkable degree of success which justifies the popularity of the Weinberg prescription.

However, the theoretical basis of the Weinberg counting is debatable and, as we will argue, far from robust. In particular we can identify two serious problems: (i) the counterterms included in the Weinberg counting are not enough as to renormalize the amplitudes (either at LO [14, 21, 22] or NLO/N<sup>2</sup>LO [23, 24]) and (ii) it is not clear whether the scattering amplitudes obey the power counting of the chiral potential as a result of the full it-

<sup>1</sup> It should be noted that lattice EFT implementations of the Weinberg counting only iterate the order  $Q^0$  piece of the interaction, while the subleading pieces generally enter as perturbations. The  $C_2(p^2 + p'^2)$  short range operator is an exception as it is promoted from order  $Q^2$  to order  $Q^0$  to form an “improved” LO interaction.

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eration process. In addition, there is the issue of the chiral inconsistency [43] which inspired the Kaplan, Savage and Wise (KSW) counting [44, 45]. Without these ingredients, renormalizability and power counting, the Weinberg prescription is merely reduced to a recipe for constructing nuclear potentials.

The renormalizability problem is a consequence of the appearance of singular interactions in the chiral expansion of the potential: the order  $\nu$ -th contribution to the potential behaves as  $V^{(\nu)}(r) \sim 1/r^{3+\nu}$  for  $m_\pi r \leq 1$ . In this regard the theory of non-perturbative renormalizability developed in Refs. [21–24, 46] states that, if the finite range potential is singular and attractive, counterterms need to be included for removing the cut-off dependence. This result implies in particular the non-renormalizability of the Weinberg counting at any order, as there is always an infinite number of partial waves for which the chiral potential is singular and attractive. Two possible solutions are (i) to treat the chiral potential, in particular one pion exchange (OPE), perturbatively in partial waves with sufficiently high angular momentum, as advocated in Refs. [21, 47], or (ii) to realize that the infinite number of counterterms is partly an artifact of the partial wave projection that can be avoided by correlating the short-range operators of different channels [28]. In the present work we will follow the first solution, which is the simplest and most natural one from the EFT perspective.

On the contrary, a repulsive singular interaction is completely insensitive to the value of the counterterms. This condition leads to unexpected consequences, in particular the deuteron NLO disaster [23]: at NLO the chiral potential turns out to be repulsive in the triplet channel leading to the disappearance of the deuteron as the cut-off is removed, in clear disagreement with the experimental evidence. In addition, the eventual change of sign of the chiral potential at certain orders prevents the formulation of a sensible power counting in a fully non-perturbative set-up, as has been recently stressed by Entem and Machleidt [48].

However, the difficulties associated with non-perturbative renormalizability are not the only reason to avoid the iteration of chiral two pion exchange (TPE). A different, but also potentially serious issue is the possible mismatch between the power counting in the potential and the physical observables. As is well known, the Weinberg prescription defines power counting in terms of the NN potential, which is not an observable. The iteration of the potential can alter the relative importance of each component of the interaction, specially if the cut-off is not soft enough [31, 32, 49, 50], and thus destroy the original power counting at the level of observables. In this respect the preliminary results of Ref. [51] for the singlet channel indicate that this may be happening for the typical values of the cut-off employed in the Weinberg scheme, calling for a perturbative reanalysis of the N<sup>2</sup>LO [15, 18, 19] and N<sup>3</sup>LO [17, 20] results.

As argued in the previous paragraphs, the combined

requirement of renormalizability and power counting imposes stringent constraints on the set of theoretically acceptable EFT formulations of the nuclear force. At leading order the non-perturbative features of the two-nucleon interaction requires the iteration of certain pieces of the interaction, minimally a contact operator in the  $^1S_0$  and  $^3S_1$  channels [44, 45, 52], and optimally the OPE potential in s- and p-waves (and eventually d-waves), plus the extra counterterms required to renormalize the scattering amplitude [21]. In order to avoid counterterm proliferation, we should be conservative with respect to what pieces of the chiral potential we iterate. Finally, to guarantee the existence of a power counting and prevent the renormalization issues we have mentioned, the sub-leading pieces of the interaction should be perturbative corrections over the LO results. The modified Weinberg scheme of Ref. [21] fulfills these conditions and is in addition a phenomenologically promising approach, as can be inferred from its lowest order results. However, as we are dealing with *a posteriori* power counting, the previous is not necessarily the only possible realization of a consistent and phenomenologically acceptable nuclear EFT, and there may be other frameworks or variations (see Refs. [53, 54] for two examples) that may eventually work as well.

The present work explores the perturbative renormalizability of chiral TPE at NLO and N<sup>2</sup>LO assuming the non-perturbative treatment of OPE at LO. This formulation is to be identified with the proposal of Nogga, Timmermans and van Kolck [21] for constructing a consistent power counting for nuclear EFT. Regarding the formal aspects of the theory, we elaborate upon the perturbative renormalization framework of Ref. [55] and determine the general conditions that ensure the renormalizability of the subleading corrections to the LO scattering amplitudes. That is, we deduce the power counting of the contact operators. As expected the scaling of the counterterms is very similar to the one previously obtained by Birse [47]. As an application of the formalism, we compute the p- and d-wave phase shifts resulting in a phenomenologically acceptable description which improves over the Weinberg counting at the same order [19].

The general approach we follow and the perturbative techniques we employ, which are based on distorted wave Born approximation (DWBA), are equivalent to the momentum space formulation of Refs. [56, 57]. In particular, the recent exploration of the  $^3P_0$  partial wave by Long and Yang [57] leads to conclusions that are equivalent to the ones obtained in the present work for the aforesaid wave. The role of perturbative chiral TPE has also been studied in the “deconstruction” approach of Refs. [58–60], which analyzes the scaling of the short range interaction for most of the uncoupled s-, p- and d-waves within the power counting derived from Ref. [21]. In addition deconstruction provides interesting insights on important aspects of the theory such as the determination of the expansion parameter of the EFT, which we will employ in the present work.

The article is structured as follows: in Sect. II we review the modifications induced by the requirement of renormalizability to the Weinberg power counting at leading [21] and subleading [55] orders. In Sect. III we explore in detail the divergences related to the perturbative treatment of chiral TPE within the DWBA to determine the power counting of the subleading contact operators at NLO and N<sup>2</sup>LO. The p- and d-waves phase shifts resulting from this renormalization scheme are presented in Sect. IV. Finally, we discuss the results and present our conclusion in Sect. V. In the appendix A we explain the technical details behind the LO non-perturbative renormalization.

## II. MODIFICATIONS TO THE WEINBERG POWER COUNTING

In this section we want to explain in an informal manner why renormalizability requires certain modifications to the Weinberg counting. The starting point is the work of Nogga, Timmermans and van Kolck [21], which pointed out, by means of a thorough numerical exploration, that cut-off independence of the leading order nucleon-nucleon scattering amplitude requires new counterterms in partial waves for which the LO tensor force is attractive. This precise pattern is naturally explained within the non-perturbative renormalization framework developed in Refs. [22–24], which proved that the inclusion of a specific number of counterterms is a necessary and sufficient condition for the renormalizability of attractive singular potentials. Curiously, the failure of the Weinberg scheme is not accidental: the existence of singular interactions in the EFT expansion of the chiral potential is just a consequence of power counting itself. We will illustrate this point in Sect. II A. For exemplifying the ideas behind the non-perturbative renormalization of singular potentials and the LO modifications of the power counting [21–24], we will consider the particular case of the <sup>3</sup>P<sub>0</sub> partial wave in Sect. II B.

Nevertheless, the purpose of the present work is to explore the extension of the Nogga et al. [21] counting to subleading orders. As already conjectured in Ref. [21], the most adequate approach for formulating a canonical EFT expansion of the amplitudes is the perturbative treatment of the higher order corrections to the chiral nuclear potential and the addition of a certain number of counterterms to guarantee the renormalizability of the approach. The power counting for the subleading contact interactions, that is, the order of the promoted counterterms, was constructed for the first time by Birse in Ref. [47] (see also Ref. [61] for a more accessible presentation). More recently, Ref. [55] proved the viability of the previous proposal and found some deviations from the original power counting of Ref. [47]. Incidentally, the subleading power counting deviates from the original Weinberg prescription even in partial waves for which the leading order counting remained unaltered in the Nogga

et al. proposal [21]. The best example is provided by the <sup>1</sup>S<sub>0</sub> partial wave: although at LO both Weinberg and Nogga et al. predict the same number of counterterms, at NLO and N<sup>2</sup>LO the counterterms predicted by Weinberg are not enough as to renormalize the scattering amplitude in this partial wave. Based on the related discussion of Ref. [51], we will explain the power counting of the singlet channel in Sect. II C.

### A. Singular Potentials

The appearance of singular interactions in the chiral potential is the fundamental reason which requires the power counting modifications suggested by Nogga et al. [21]. Curiously, singular potentials are a consequence of the existence of a power counting in the potential itself. The argument for the previous statement is relatively straightforward. We first consider the scaling of the order  $\nu$ -th contribution to the chiral potential, which according to power counting is given by

$$V^{(\nu)} \propto \frac{Q^\nu}{\Lambda_0^\nu}, \quad (2)$$

where  $Q$  represents the light scales of the system, like the pion mass or the relative momentum of the nucleons, while  $\Lambda_0$  stands for the heavy scales, for example the mass of the  $\rho$  meson.

It is interesting to notice that the two aforementioned light scales play a very different role: while the pion mass  $m_\pi$  is fixed, the momentum of the nucleons can vary, and in particular it can take values that are much larger than the pion mass, that is,  $|\vec{q}| \gg m_\pi$ . In this regime the momentum space representation of  $V^{(\nu)}$  fulfills the approximate scaling law

$$V^{(\nu)}(\lambda \vec{q}) \propto \lambda^\nu V^{(\nu)}(\vec{q}), \quad (3)$$

which is just a restatement of Eq. (2). After Fourier transforming, this translates into the following scaling relation in coordinate space

$$V^{(\nu)}(\lambda \vec{r}) \propto \frac{V^{(\nu)}(\vec{r})}{\lambda^{\nu+3}}, \quad (4)$$

which is valid for  $m_\pi |\vec{r}| \ll 1$ .

There are two kinds of solution to the scaling relations induced by the power counting of the potential: (i) zero range and (ii) finite range solutions. The zero (or contact) range solutions are constructed from the scaling properties of the three dimensional Dirac delta, that is

$$\delta(\lambda \vec{r}) = \frac{\delta(\vec{r})}{\lambda^3}, \quad (5)$$

which corresponds to a solution of the scaling equation for  $\nu = 0$ , yielding

$$V_C^{(0)}(\vec{r}) = C_0 \delta(\vec{r}), \quad (6)$$

where the subscript  $C$  indicates the contact range nature of the contribution. Higher order contact range solutions

can be easily obtained by adding derivatives to the Dirac delta, although due to parity conservation only an even number of derivatives can appear. That is, new contact terms enter only at even orders,  $\nu = 2n$ , as is well known.

The finite range solutions of the scaling relations are fairly straightforward and take the form

$$V_F^{(\nu)}(r) \propto \frac{1}{r^{3+\nu}}, \quad (7)$$

where the subscript  $F$  is used to indicate the finite range. This form corresponds to the observed degree of divergence of the pion contributions to the chiral potentials<sup>2</sup>. As we will see, the renormalizability problems arise from the non-perturbative interplay between contact and finite range operators, which modifies their scaling properties at the level of observables, thus changing the power counting of the operators.

## B. Leading Order Modifications of the Counting

Weinberg power counting implicitly assumes that the counterterms appearing at each order in the chiral expansion are capable of renormalizing the scattering amplitude. However, this is not the case, as has been repeatedly discussed in the literature [21, 23, 24, 43]. In particular, Nogga et al. [21] found numerically that even at leading order the counterterms of the Weinberg counting do not render the phase shifts unique: several counterterms should be added in the  ${}^3P_0$ ,  ${}^3P_2 - {}^3F_2$  channels and eventually the  ${}^3D_2$  channel, depending on the cut-off region under consideration (see also the related comments of Ref. [63])<sup>3</sup>.

The  ${}^3P_0$  channel probably provides the best example to exemplify the ideas behind the Nogga et al. proposal [21]. We start by consider this partial wave in the Weinberg counting, where we can explicitly check the appearance of a strong cut-off dependence of the  ${}^3P_0$  phase shift at LO. In the Weinberg scheme, the wave function in this channel can be described by the following Schrödinger equation

$$-u_k^{(0)''} + \left[ 2\mu V_{3P_0}^{(0)}(r) + \frac{2}{r^2} \right] u_k(r) = k^2 u_k^{(0)}(r), \quad (8)$$

where  $V_{3P_0}^{(0)}(r)$  is the order  $Q^0$  chiral potential. As the potential diverges as  $1/r^3$  at short distances, we include a regularization scale  $r_c$  which serves as a separation scale between the unknown short range physics and the known

long range physics. In particular, we only consider the chiral potential to be valid for  $r > r_c$ . For radii below the regularization scale,  $r < r_c$ , we do not know how the system can be described, but it is reasonable to assume that at short enough distances the behaviour of the wave function will be dominated by the centrifugal barrier, that is,

$$u_k^{(0)}(r) \simeq a r^2, \quad (9)$$

for  $r < r_c$ . The previous assumption can be effectively translated into a logarithmic boundary condition for the Schrödinger equation at  $r = r_c$

$$\left. \frac{u_k'(r)}{u_k(r)} \right|_{r=r_c} \simeq \frac{2}{r_c}, \quad (10)$$

corresponding to the radial regulator employed in Ref. [24]. By integrating the Schrödinger equation from  $r = r_c$  to  $r \rightarrow \infty$  with the previous initial integration condition, we can extract the phase shifts by matching to the asymptotic form of the wave function at large distances, which is given by

$$u_k^{(0)}(r) \rightarrow \cot \delta_{3P_0}^{(0)} \hat{j}_1(kr) - \hat{y}_1(kr), \quad (11)$$

where  $\delta_{3P_0}^{(0)}$  is the LO  ${}^3P_0$  phase shift, and  $\hat{j}_1(x) = x j_1(x)$ ,  $\hat{y}_1(x) = x y_1(x)$ , with  $j_1(x)$  and  $y_1(x)$  the Spherical Bessel functions.

The results for the  ${}^3P_0$  phase shifts in the previous regularization scheme are depicted in Fig. (1). The phase shifts, which have been computed for a series of cut-off radii ranging from  $r_c = 0.6$  fm to 1.6 fm, show a remarkable cut-off dependence not anticipated by the Weinberg counting. The formal reason for this cut-off dependence lies in the details of the short range ( $m_\pi r \ll 1$ ) solution to the Schrödinger equation Eq. (8), which reads

$$u_k^{(0)}(r) \propto (ka_3) \left( \frac{r}{a_3} \right)^{3/4} \sin \left[ \sqrt{\frac{a_3}{r}} + \varphi \right], \quad (12)$$

where  $a_3$  is a length scale related to the strength of the tensor force in this channel. As can be seen, the determination of the wave function near the origin requires the determination of the phase  $\varphi$ . The boundary condition, Eq. (10), just chooses a different, non-converging value of  $\varphi$  for each  $r_c$ , and therefore does not yield unique results. In particular, there is not a well defined limit of  $\varphi$  for  $r_c \rightarrow 0$ : the value of  $\varphi$  simply oscillates faster and faster on the way to the origin.

From the EFT point of view, the previous cut-off dependence means that the a priori estimation of the strength of the contact term for the  ${}^3P_0$  channel was not correct. Otherwise, the cut-off dependence would have not appeared until radii of the order of the breakdown scale of the theory  $\Lambda_0 r_c \sim 1$ . However, as can be seen in the left panel of Fig. (1), the cut-off dependence manifest for scales of the order of  $m_\pi r_c \sim 1$ . This indicates the necessity of including a new short range operator in this channel at LO.

<sup>2</sup> The inclusion of light but static degrees of freedom, like the  $\Delta$  isobar in the small scale expansion [62], can alter however the previous scaling.

<sup>3</sup> Notice however that the recent application of the N/D method to nuclear EFT [33] does not require the inclusion of an additional counterterm in the  ${}^3P_0$  or the  ${}^3D_2$  partial waves.

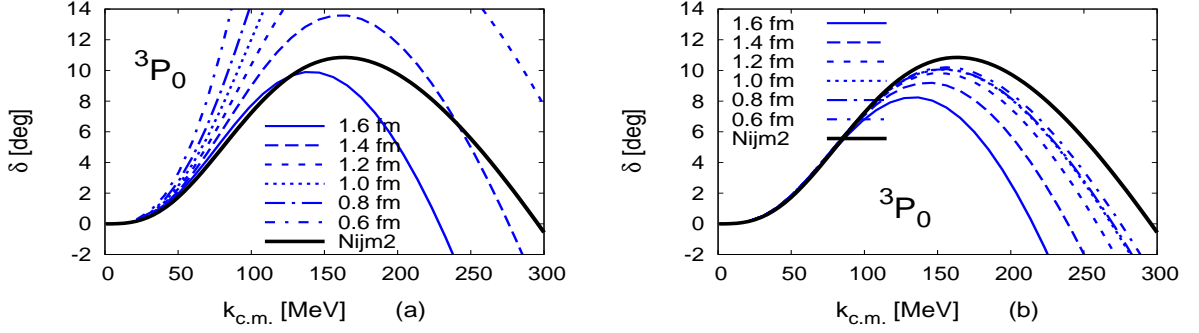


FIG. 1. (Color online) Leading order phase shifts for the  $^3P_0$  channel in the Weinberg counting (left panel) and in the modified Weinberg scheme of Nogga, Timmermans and van Kolck [21] (right panel). As can be seen, the  $^3P_0$  phases develop a strong cut-off dependence in the Weinberg counting, which can be eliminated by the promotion of a counterterm as advocated in Ref. [21].

By including the new counterterm, the previous uncontrolled cut-off dependence of the  $^3P_0$  phase shift disappears, as can be appreciated in the right panel of Fig. (1). We have employed the regularization scheme of Refs. [22–24], in which to unambiguously determine the LO phase shifts in the  $^3P_0$  partial wave we fix the value of the scattering length<sup>4</sup>, see the appendix A for the details. Of course, there is still a residual cut-off dependence that is however mostly harmless owing to the existence of a clear convergence pattern in the  $r_c \rightarrow 0$  limit. That is, we can interpret the residual cut-off dependence as a higher order effect. For the regularization employed in this case, the cut-off dependence of the phase shifts can be explicitly computed by employing the methods of Ref. [64], yielding the result

$$\frac{d\delta_{^3P_0}^{(0)}(k; r_c)}{dr_c} = k^3 \left| \frac{u_k^{(0)}(r_c)}{k} \right|^2 \sim k^3 r_c^{3/2} a_3^{1/2}, \quad (13)$$

which serves as a formal confirmation of the existence of the  $r_c \rightarrow 0$  limit.

### C. Subleading Order Modifications of the Counting

The modifications of the power counting induced by renormalizability are not confined to leading order. Further deviations occur at subleading orders, which can be identified by analyzing the cut-off dependence of the

phase shifts, even though these orders are included as perturbations. In this case, the most direct example is the  $^1S_0$  channel, which follows the Weinberg counting at LO, but diverts from it when subleading corrections are included. At orders  $Q^2$  and  $Q^3$  the Weinberg counting dictates a total of two counterterms for the  $^1S_0$  channel, which in momentum space take the form

$$\langle p | V_C^{(\nu=2,3)} | p' \rangle = C_0^{(\nu)} + C_2^{(\nu)}(p^2 + p'^2). \quad (14)$$

In terms of renormalization, this counterterm structure is equivalent to fixing two observables, for example the scattering length and the effective range of the  $^1S_0$  channel. For making such a calculation we will follow the formalism of Ref. [55], but adapting the number of counterterms to two. In such a case, we obtain the phase shifts of Fig. (2).

As can be appreciated, there is a strong cut-off dependence in the results of Fig. (2). The reason can be found by analyzing the cut-off behaviour of the perturbative corrections to the phase shifts. According to Ref. [55], the  $Q^\nu$  contribution to the phase shift is proportional to the integral

$$\delta^{(\nu)}(k; r_c) \propto \frac{\mu}{k} \int_{r_c}^{\infty} dr V^{(\nu)}(r) u_k^{(0)2}(r), \quad (15)$$

where  $u_k^{(0)}(r)$  is the LO wave function in the Weinberg counting, which can be described by adapting the Schrödinger equation for the  $^3P_0$  channel, Eq. (10), to the  $^1S_0$  case, and  $V^{(\nu)}$  is the order  $Q^\nu$  contribution to the chiral potential. It should be noted that  $\delta^{(\nu)}$  scales as  $Q^{\nu+1}$ , even though nominally it is of order  $Q^\nu$ . The degree of divergence of  $\delta^{(\nu)}$  can be easily determined by considering the Taylor expansion of the LO wave function [23], that is

$$u_k^{(0)}(r) = u_0 + k^2 u_2 + k^4 u_4 + \dots, \quad (16)$$

where the behaviour of the different terms at short enough distances is given by  $u_0 \sim 1$ ,  $u_2 \sim r^2$ ,  $u_4 \sim r^4$ ,

<sup>4</sup> Of course, using the term scattering *length* for a  $p$ -wave is language abuse. The proper name is scattering volume, as it has dimensions of [length]<sup>3</sup>. A similar comment would apply in the  $d$ -wave case, in which the so-called scattering *length* has dimension of [length]<sup>5</sup> and henceforth is a (5-dimensional) hypervolume. However, for notational simplicity we will always use the term scattering length for the coefficient related to the low energy behaviour of the  $l$ -wave phase shift,  $\delta_l(k) = -a_l k^{2l+1} + \mathcal{O}(k^{2l+3})$ .

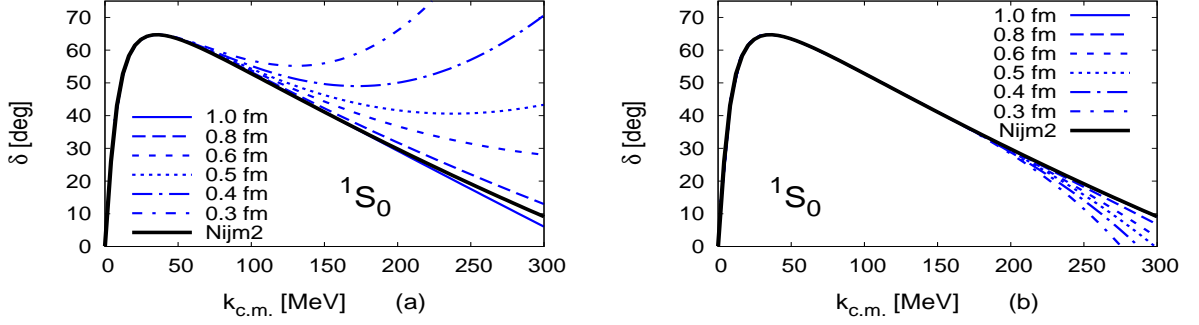


FIG. 2. (Color online) Next-to-next-to leading order phase shifts for the  $^1S_0$  channel in the perturbative Weinberg counting (left panel) and in the Nogga, Timmermans and van Kolck scheme [21] (right panel). By perturbative Weinberg we mean that the subleading pieces of the potential (that is, chiral two pion exchange) are treated as perturbations. We can appreciate that the cut-off dependence appearing in the perturbative Weinberg set-up can be cured by the inclusion of an additional counterterm.

and so on. With the previous information, and the additional fact that  $V^{(\nu)} \sim 1/r^{3+\nu}$ , we find that for the particular case  $\nu = 3$  the integral in Eq. (15) diverges as

$$\begin{aligned} I^{(\nu=3)}(k; r_c) &= \int_{r_c}^{\infty} dr V^{(\nu)}(r) u_k^{(0)2}(r) \\ &= \left(\frac{a}{r_c}\right)^5 c_0 + (ka)^2 \left(\frac{a}{r_c}\right)^3 c_2 \\ &\quad + (ka)^4 \left(\frac{a}{r_c}\right) c_4 + \dots, \end{aligned} \quad (17)$$

with  $a$  the length scale governing the divergences and where  $c_0$ ,  $c_2$  and  $c_4$  are dimensionless numbers that are expected to be of order one, while the terms in the dots are already finite. In Fig. (2) we have only fixed the  $k^0$  and  $k^2$  behaviour for reproducing the low energy phase shifts, that is, the scattering length and the effective range. This means that there is still a contribution that diverges as  $k^4/r_c$  explaining the failure of the Weinberg counting in the perturbative context.

Again, the solution is to modify the counting in such a way as to include a new counterterm. This means that the correct  $Q^2$  and  $Q^3$  contact interaction should be <sup>5</sup>

$$\begin{aligned} \langle p | V_C^{(\nu=2,3)} | p' \rangle &= C_0^{(\nu)} + C_2^{(\nu)}(p^2 + p'^2) \\ &\quad + C_4^{(\nu)}(p^4 + p'^4), \end{aligned} \quad (18)$$

<sup>5</sup> We have ignored the  $C_4^{(\nu)} p^2 p'^2$  operator, as it is redundant [65]. However, the equivalence of this operator with  $C_4(p^4 + p'^4)$  only shows up either in dimensional regularization or once the floating cut-off is removed. For finite cut-off calculations, the difference between the  $C_4$  and  $C_4'$  operators is presumably a higher order effect. On the practical side the inclusion of this operator will surely be advantageous in momentum space treatments as it may help to reduce the cut-off dependence of the scattering observables.

instead of the form dictated by Weinberg dimensional counting in Eq. (14). The additional counterterm is able to absorb the  $k^4/r_c$  divergence in the N<sup>2</sup>LO phase shift. In such a case, the new cut-off dependence is the one given by Fig. (2), in which there is still a residual cut-off dependence of  $\mathcal{O}(k^6 r_c)$ . However, the same comments apply as in the  $^3P_0$  channel at LO: the residual cut-off dependence is a higher order effect and can be safely ignored.

Of course, the power counting modification of Eq. (18) is well-known to arise in two-body systems with a large value of the scattering length [44, 45, 66, 67], where the renormalization group analysis of the  $C_4$  operator indicates that its actual scaling is  $Q^2$  rather than the naive dimensional expectation  $Q^4$ . In this regard, it should be stressed that the  $^1S_0$  phase shifts cannot be properly renormalized at NLO/N<sup>2</sup>LO without the inclusion of the  $C_4$  operator. The previous is not merely a statement about the existence of the  $r_c \rightarrow 0$  limit in the calculation, but also about the accuracy of the theoretical description of the system. The renormalized calculation is accurate up to  $Q^3(Q^4)$  at NLO (N<sup>2</sup>LO). On the contrary, if we do not include the  $C_4$  operator, the error of the calculation will be  $\mathcal{O}(Q^2)$ , much bigger than expected. This may prove to be a significant drawback for the nuclear EFT lattice approach [36–42], which does not include the aforementioned operator in the  $^1S_0$  channel.

Finally, the observation that the contact interaction of Eq. (18) renormalizes the scattering amplitude can be employed to extract the breakdown scale of the EFT in the singlet channel, as has been shown by the deconstruction approach of Refs. [58, 59]. The idea is to employ the phenomenological phase shifts as the input of a calculation for determining the form of the contact range interaction up to an error of  $\mathcal{O}(Q^{\nu+1})$ . This represents in fact the complementary process of fitting the  $C_{2n}^{(\nu)}$  parameters to the phase shifts. By considering the on-shell form of the contact range potential,  $V_C^{(\nu)}(k) = \langle k | V_C^{(\nu)} | k \rangle$ ,

Ref. [59] is able to determine that (i) the form of the short range interaction complies with Eq. (18) and (ii) the hard scale governing  $V_C$  is  $\Lambda_{0,s} \simeq 270$  MeV. This figure is much lower than the expected  $\Lambda_0 \sim 0.5$  GeV and translates into a slow convergence rate for the singlet channel,  $Q/\Lambda_0 \simeq m_\pi/\Lambda_{0,s} \simeq 0.5$ , emphasizing the importance of the proper renormalization of the scattering amplitude to minimize the theoretical uncertainty. The breakdown scale can in fact be appreciated in the right panel of Fig. (2), where the cut-off dependence of the renormalized results starts to become sizable at  $k \simeq \Lambda_{0,s}$ , in agreement with the estimates of Ref. [59]<sup>6</sup>.

### III. PERTURBATIVE RENORMALIZABILITY OF SINGULAR POTENTIALS AND POWER COUNTING

In this section we study the power counting that arises when one pion exchange is iterated to all orders and chiral two-pion exchange is treated as a perturbation. We follow a more formal and detailed approach than in the previous section. We start by defining the power counting conventions in Sect. III A. We derive the power counting by requiring the perturbative corrections to remain finite in the  $r_c \rightarrow 0$  limit, first for the uncoupled channels in Sect. III B and then we extend the results to the coupled channels in Sect. III C. We informally comment on the convergence rate and the expansion parameter of the resulting EFT in Sect. III D. Finally, we discuss the perturbative power counting in Sect. III E. The results of this section are briefly summarized in Table I, where we can check the number of counterterms required in the lower partial waves from LO up to N<sup>3</sup>LO.

#### A. Conventions

We assume that the chiral potential is expanded in terms of a power counting as follows

$$V(\vec{q}) = \sum_{\nu=\nu_0}^{\nu_{\max}} V^{(\nu)}(\vec{q}) + \mathcal{O}\left(\left(\frac{Q}{\Lambda_0}\right)^{\nu_{\max}+1}\right), \quad (19)$$

where  $Q$  ( $\Lambda_0$ ) is the soft (hard) scale of the system. The expansion starts at order  $\nu_0 \geq -1$ , and we only consider

the corrections to the chiral potential up to a certain order  $\nu_{\max}$  ( $= 2, 3$  in the present paper). The T-matrix follows the same power counting expansion as the potential, that is

$$T = \sum_{\nu=\nu_0}^{\nu_{\max}} T^{(\nu)} + \mathcal{O}\left(\left(\frac{Q}{\Lambda_0}\right)^{\nu_{\max}+1}\right). \quad (20)$$

The scattering equations can be obtained by reexpanding the Lippmann-Schwinger equation

$$T = V + VG_0T, \quad (21)$$

in terms of the power counting of the potential, taking into account that the resolvent operator  $G_0$  is formally of order  $Q$ . In this convention, only contributions to the chiral potential which are of order  $Q^{-1}$  should be iterated, yielding the equation

$$T^{(-1)} = V^{(-1)} + V^{(-1)}G_0T^{(-1)}. \quad (22)$$

The corresponding dynamical equation of the higher order contributions to the T-matrix are increasingly more involved.

In principle, in the Weinberg scheme the power counting expansion starts at order  $Q^0$

$$V_{\text{NN}} = V^{(0)} + V^{(2)} + V^{(3)} + \mathcal{O}(Q^4). \quad (23)$$

Within the power counting conventions we follow, the previous means that all pieces of the potential should be treated as perturbations. However, it is clear that  $V^{(0)}$  needs to be iterated in certain partial waves. This requires a promotion from order  $Q^0$  to  $Q^{-1}$ , that is

$$V^{(0)} \rightarrow V^{(-1)}, \quad (24)$$

at least in the partial waves in which either the LO contact operator or OPE are non-perturbative. From a purely quantum-mechanical point of view the iteration is necessary if the potential is not weak. In terms of power counting, a large coupling constant can be accounted for by identifying it with  $\Lambda_0/Q$ , which is a large number. Alternatively, we can try to explicitly identify the low energy scale which requires the LO potential to be iterated. For the LO contact operator the additional low energy scale is well known to be the inverse of the large scattering length of the  $^1S_0$  and  $^3S_1 - ^3D_1$  channels, as has been extensively discussed in the literature [44, 45, 52, 66, 68]. For justifying the iteration of the OPE potential Birse has proposed [47] the identification of

$$\Lambda_{\text{OPE}} = \Lambda_{\text{NN}} = \frac{16\pi f_\pi^2}{M_N g_A^2} \simeq 300 \text{ MeV} \quad (25)$$

as the new low energy scale. However, this choice does not explain the fact that OPE remains perturbative in the singlet channels even if iterated [69]. A different possibility may be to treat the inverse of the length scale governing the strength of the tensor force as the missing low energy scale, giving a  $\Lambda_{\text{OPE}}$  that is in general a fraction of  $\Lambda_{\text{NN}}$ .

<sup>6</sup> A similar comment applies for the phase shifts computed in the Weinberg counting: if we inspect the left panels of Figs. (1) and (2), the cut-off dependence is already conspicuous at  $k \simeq 100 - 200$  MeV, suggesting a relatively soft breakdown scale for both the  $^1S_0$  and  $^3P_0$  partial waves. In contrast, for the  $^3P_0$  channel in the Nogga et al. counting [21] the cut-off dependence of the right panel of Fig. (1) does not provide a clear indication of the breakdown scale. In this case the determination of the range of applicability of the theory is better done by analyzing the size the subleading order corrections to the phase shift.

| Partial wave  | LO | NLO | N <sup>2</sup> LO | N <sup>3</sup> LO |
|---|----|-----|-------------------|-------------------|
| <sup>1</sup> S <sub>0</sub>                               | 1  | 3   | 3                 | 4                 |
| <sup>3</sup> S <sub>1</sub> – <sup>3</sup> D <sub>1</sub> | 1  | 6   | 6                 | 6                 |
| <sup>1</sup> P <sub>1</sub>                               | 0  | 1   | 1                 | 2                 |
| <sup>3</sup> P <sub>0</sub>                               | 1  | 2   | 2                 | 2                 |
| <sup>3</sup> P <sub>1</sub>                               | 0  | 1   | 1                 | 2                 |
| <sup>3</sup> P <sub>2</sub> – <sup>3</sup> F <sub>2</sub> | 1  | 6   | 6                 | 6                 |
| <sup>1</sup> D <sub>2</sub>                               | 0  | 0   | 0                 | 1                 |
| <sup>3</sup> D <sub>2</sub>                               | 1  | 2   | 2                 | 2                 |
| <sup>3</sup> D <sub>3</sub> – <sup>3</sup> G <sub>3</sub> | 0  | 0   | 0                 | 1                 |
| All   | 5  | 21  | 21                | 27                |

TABLE I. Total number of counterterms per partial wave at LO ( $Q^{-1}$ ), NLO ( $Q^2$ ), N<sup>2</sup>LO ( $Q^3$ ) and N<sup>3</sup>LO ( $Q^4$ ) assuming (i) the iteration of the OPE potential at LO in the <sup>3</sup>S<sub>1</sub> – <sup>3</sup>D<sub>1</sub>, <sup>3</sup>P<sub>0</sub>, <sup>3</sup>P<sub>2</sub> – <sup>3</sup>F<sub>2</sub> and <sup>3</sup>D<sub>2</sub> partial waves and (ii) energy-dependent counterterms. The N<sup>3</sup>LO results assume that an energy-independent representation of the finite-range chiral potential is being used. If an energy-dependent potential is employed, the number of free parameters will grow from 27 to 35 (see the discussion in Sect.III E for details).

After the promotion of LO potential from order  $Q^0$  to order  $Q^{-1}$ , we encounter a remarkable simplification in the dynamical equations describing the  $Q^2$  and  $Q^3$  contributions to the T-matrix, namely

$$T^{(\nu)} = (1 + T^{(-1)}G_0)V^{(\nu)}(G_0T^{(-1)} + 1), \quad (26)$$

where, due to the absence of contributions to the finite range chiral potential from order  $Q^{-1}$  to order  $Q^2$ , second order perturbation theory is not needed up to order  $Q^5$ . That is, the equation above is valid for  $\nu < 5$ .

## B. Uncoupled Channels

We consider in the first place the power counting expansion of the  $l$ -wave phase shift

$$\delta_l(k; r_c) = \sum_{\nu=-1}^{\nu_{\max}} \delta_l^{(\nu)}(k; r_c) + \mathcal{O}(Q^{\nu_{\max}+1}). \quad (27)$$

The LO phase shift is calculated non-perturbatively by solving the reduced Schrödinger equation with the LO chiral potential, see Appendix A for details. On the other hand, we compute the order  $\nu$ -th contribution to the phase shift,  $\delta_l^{(\nu)}$ , in the distorted wave Born approximation (DWBA), that is

$$\frac{\delta_l^{(\nu)}(k; r_c)}{\sin^2 \delta_l^{(-1)}} = -\frac{2\mu}{k^{2l+1}} \mathcal{A}_l^{(-1)2} I_l^{(\nu)}(k; r_c) \quad (28)$$

where  $\delta_l^{(-1)}$  is the leading order phase shift,  $\mu$  the reduced mass of the two nucleon system,  $k$  the center-of-mass momentum,  $l$  the angular momentum of the system,  $\mathcal{A}_l^{(-1)}$

a normalization factor and  $I_l^{(\nu)}$  the perturbative integral. The perturbative integral  $I_l^{(\nu)}$  is defined as

$$I_l^{(\nu)}(k; r_c) = \int_{r_c}^{\infty} dr V^{(\nu)}(r) u_{k,l}^{(-1)}(r)^2 \quad (29)$$

where  $u_{k,l}^{(-1)}$  is the LO wave function, which we assume to be asymptotically ( $r \rightarrow \infty$ ) normalized to

$$\frac{\mathcal{A}_l^{(-1)}}{k^l} u_{k,l}^{(-1)}(r) \rightarrow \cot \delta_l^{(-1)} \hat{j}_l(kr) - \hat{y}_l(kr), \quad (30)$$

where  $\hat{j}_l(x) = x j_l(x)$ ,  $\hat{y}_l(x) = x y_l(x)$ , with  $j_l(x)$  and  $y_l(x)$  the spherical Bessel functions. For the purpose of analyzing the renormalization of the perturbative phase shifts, we will assume that the normalization factor  $\mathcal{A}_l^{(-1)}(k)$  is defined in such a way that the LO wave function is energy-independent at  $r \rightarrow 0$ , that is

$$\lim_{r \rightarrow 0} \frac{d^2}{dk^2} u_{k,l}(r) = 0, \quad (31)$$

a prescription that will make relatively easy the identification of the necessary number of counterterms. However, in practical calculations we will simply take

$$\left. \frac{d^2}{dk^2} u_{k,l}^{(-1)}(r) \right|_{r=r_c} = 0, \quad (32)$$

which is easier to implement than Eq. (31).

The renormalizability of the perturbative phase shifts can be easily analyzed in terms of the short range behaviour of the LO reduced wave function and the  $Q^\nu$  contribution to the chiral potential, which determine whether the perturbative integral  $I_l^{(\nu)}$  is divergent or not. In particular, for  $mr \ll 1$ , we have the following

$$u_{k,l}^{(-1)}(r) \sim r^s, \quad V^{(\nu)}(r) \sim \frac{1}{r^{3+\nu}}, \quad (33)$$

where  $s$  describes the power-law behaviour of the wave function at short distances. For the perturbative integral we have

$$I_l^{(\nu)}(k, r_c) \sim \int_{r_c}^{\infty} \frac{dr}{r^{3+\nu-2s}}, \quad (34)$$

which diverges if  $3 + \nu - 2s \geq 1$ . The necessary number of counterterms can be determined by considering the Taylor expansion of the perturbative integral in terms of  $k^2$

$$I_l^{(\nu)}(k, r_c) = \sum_{n=0}^{\infty} I_{2n,l}^{(\nu)}(r_c) k^{2n}, \quad (35)$$

in which each new term is less singular than the previous one as a consequence of the related expansion for the LO reduced wave function

$$u_{k,l}^{(-1)}(r) = \sum_{n=0}^{\infty} u_{2n,l}^{(-1)}(r) k^{2n}, \quad (36)$$



where  $u_{2n,l}^{(-1)} \sim r^{s+nt}$  for small radii, with  $t \geq 2$ . We can see now that the reason for choosing the energy-independent normalization of Eq. (31) is to have additional power law suppression at short distances in the  $k^2$  expansion of the wave function. In terms of the expansion of the perturbative integral, we obtain

$$I_{2n,l}^{(\nu)}(r_c) \sim \int_{r_c} \frac{dr}{r^{3+\nu-2s-nt}}, \quad (37)$$

which converges for  $3 + \nu - 2s - nt < 1$ . Eventually, as  $n$  increases, we will have well-defined integrals. In this regard, we can define the number of subtraction/counterterms as the smallest value of  $n_c$  for which

$$3 + \nu - 2s - n_c t < 1. \quad (38)$$

It should be noted that  $n_c$  refers to the total number of subtraction/counterterms required at a given order. For example, if  $n_c = 1$  the counterterm only affects the scattering length of the  $\nu$ -th order of the phase shift. If the scattering length was already fixed at a lower order, this just indicates that a perturbative correction must be added to the lower order counterterm.

To summarize, we have that the perturbative integral can be divided into a divergent and a regular piece

$$I_l^{(\nu)}(k; r_c) = I_{l,D}^{(\nu)}(k; r_c) + I_{l,R}^{(\nu)}(k; r_c), \quad (39)$$

where the divergent piece can be expanded in powers of the squared momentum  $k^2$ ,

$$I_{l,D}^{(\nu)}(k; r_c) = \sum_{n=0}^{n_c-1} I_{l,2n}^{(\nu)}(r_c) k^{2n}, \quad (40)$$

with  $n_c$  the number of counterterms that renders the perturbative phase shifts finite. The perturbative integral can be regularized in any specific way which we find convenient. A particularly simple and straightforward method is to add  $n_c$  free parameters to the original integral as follows

$$\hat{I}_l^{(\nu)}(k; r_c) = \sum_{n=0}^{n_c-1} \lambda_{l,2n}^{(\nu)} k^{2n} + I_l^{(\nu)}(k; r_c), \quad (41)$$

where the parameters  $\lambda_{l,2n}^{(\nu)}$  are to be fitted to the experimental phase shifts. The previous parameters, which we will informally call  $\lambda$ -parameters, can be related to the more standard counterterms if we consider a concrete representation of the short range physics, for example

$$V_{C,l}^{(\nu)}(r; r_c) = \frac{f_l^2(r_c)}{4\pi r_c^2} \sum_{n=0}^{n_c-1} C_{2n,l}^{(\nu)}(r_c) k^{2n} \delta(r - r_c), \quad (42)$$

where  $f_l(r_c) = \frac{(2l+1)!!}{r_c^l}$  is a factor which ensures that the matrix elements of the previous potential in momentum space behaves as

$$\langle p | V_{C,l} | p' \rangle \rightarrow p^l p'^l \sum_n C_{2n,l}^{(\nu)}(r_c) k^{2n}, \quad (43)$$

for  $r_c \rightarrow 0$ . If we have chosen to work in the practical normalization of Eq. (32), the following relationship is obtained between the fitting parameters  $\lambda_{2n,l}^{(\nu)}$  and the usual counterterms

$$\lambda_{2n,l}^{(\nu)} = \frac{f_l^2(r_c)}{4\pi r_c} C_{2n,l}^{(\nu)}(r_c) u_{0,l}^{(-1)}(r_c). \quad (44)$$

For other representations of the short range physics, the relationship will take a more complex form.

We can distinguish four cases: (i) the irregular solution of a non-singular potential ( $^1S_0$ ), (ii) the regular solution of a non-singular potential ( $^1P_1$ ,  $^1D_2$ ), (iii) the general solution of an attractive singular potential ( $^3P_0$ ,  $^3D_2$ ) and (iv) the regular solution of a repulsive singular potential ( $^3P_1$ ). In the following paragraphs we will discuss each of these cases in detail.

### 1. Non-Singular Potential

Non-singular (or regular) potentials are potentials for which the regularity condition is fulfilled

$$\lim_{r \rightarrow 0} r^2 V(r) = 0. \quad (45)$$

The regularity condition implies in turn that the short range behaviour of the wave function is determined by the centrifugal barrier, which overcomes the potential at short distances. For  $r \rightarrow 0$ , we can define a regular and irregular solutions which behave as

$$u_{l,k,\text{reg}}^{(-1)}(r) \sim r^{l+1}, \quad (46)$$

$$u_{l,k,\text{irr}}^{(-1)}(r) \sim \frac{1}{r^l}. \quad (47)$$

If there is no short range physics at LO, the regular solution is chosen. On the contrary, if a contact operator is included at LO, the LO reduced wave function will be a superposition of the regular and irregular solutions.

The regular solution gives rise to the Weinberg power counting unaltered. If we consider the  $k^2$  expansion of the regular solution, Eq. (36), we have

$$u_{l,k,\text{reg}}^{(-1)} \sim r^{l+2n+1}, \quad (48)$$

that is,  $s = l + 1$  and  $t = 2$ . The convergence of the perturbative integral requires in this case

$$\nu < 2l + 2n_c, \quad (49)$$

where  $n_c$  stands for the number of counterterms in a given partial wave. Therefore new counterterms appear as expected in naive dimensional analysis, that is

$$C_{2n,l} \sim Q^{2l+2n}. \quad (50)$$

For the particular case of the  $^1P_1$  channel, the first counterterm ( $C_{0,^1P_1}$ ) is required at  $Q^2$ , a second ( $C_{2,^1P_1}$ ) at  $Q^4$ , and so on.

In contrast, the occurrence of the irregular solution at LO will change the power counting. The  $k^2$  expansion of the wave function yields in this case

$$u_{l,k,\text{irr}}^{(-1)} \sim r^{-l+2n}, \quad (51)$$

which implies  $s = -l$  and  $t = 2$ . The finiteness condition reads

$$2n_c > \nu + 2l + 2, \quad (52)$$

which implies a significant deviation from naive dimensional analysis. In fact the scaling of the counterterms is given by

$$C_{2n,l} \sim Q^{2n-2l-2}. \quad (53)$$

For the  $^1S_0$  channel, the previous requires that the first perturbative correction to the  $C_{0,1S_0}$  counterterm enters at order  $Q^{-2}$ , while the  $C_{2n,1S_0}$  counterterms (with  $n > 0$ ) are required at order  $Q^{2n-2}$ . It should be noticed that the order  $Q^{-2}$  assignment to the first perturbative correction to the  $C_0$  counterterm is merely formal: this correction actually enters at the same order as the first perturbative correction to the finite range piece of the potential, which happens at order  $Q^2$ . In fact, the  $Q^{-2}$  scaling is rather an indication of the fine-tuning required for having an unnatural scattering length. In terms of Birse's renormalization group analysis (RGA) [66], the  $Q^{-2}$  eigenvalue is a reflection of the unstable nature of the infrared fixed point associated with two-body systems with a large scattering length.

It is interesting to notice that the extension of the power counting for irregular solutions to  $p$ -waves leads to the same conclusions as  $p$ -wave Halo EFT [70], namely, that the first two contact operators should be iterated. The previous arguments imply that for a  $p$ -wave, the first counterterm will be of order  $Q^{-4}$ , the second  $Q^{-2}$ , and the third  $Q^0$ . This has two interpretations: first, that the unstable infrared fixed point is even more fine-tuned than in  $s$ -waves, and second, that it may be necessary to iterate two counterterms instead of only one. However, for the particular case of nucleon-nucleon scattering, we do not encounter this situation.

## 2. Singular Potential

The appearance of power-law singular interactions at LO entails substantial changes to the power counting of the short range operators. In particular we are interested in the effect of tensor OPE, which at short distances ( $r \rightarrow 0$ ) behaves as

$$2\mu V_{\text{tensor}}^{(-1)}(r) \rightarrow \pm \frac{a_3}{r^3}, \quad (54)$$

where  $\mu$  is the reduced mass of the two-nucleon system and  $a_3$  a length scale that governs the strength of the tensor force. We have chosen a convention in which  $a_3 > 0$  and where the attractive or repulsive character of the

potential is indicated by the explicit  $\pm$  sign. The renormalization of singular potentials implies that attractive singular interactions require the inclusion of a counterterm, while repulsive singular interactions do not [22–24], a feature which will be reviewed in the following paragraph.

For a tensor force, the terms of the  $k^2$  expansion of the wave function behave for  $r \rightarrow 0$  as

$$u_{l,2n}^{(-1)}(r) \simeq r^{3/4+5n/2} f(2\sqrt{\frac{a_3}{r}}). \quad (55)$$

If the tensor force is attractive, the function  $f$  takes the form

$$f_A(x) \sim C_S \sin x + C_C \cos x, \quad (56)$$

with  $x = 2\sqrt{\frac{a_3}{r}}$  and where the specific linear combination of sine and cosine factors is determined by the LO counterterm. In contrast, for a repulsive tensor force we have a regular and irregular solution. We have indeed that

$$f_R(x) \sim C_{\text{reg}} e^{-x} + C_{\text{irr}} e^{+x}. \quad (57)$$

where the  $e^{-x}$  solution is regular at the origin, while the  $e^{+x}$  solution diverges exponentially for  $r \rightarrow 0$ . For the repulsive case we will always chose the regular solution at LO.

In the attractive case, the convergence of the perturbative integrals is solely determined by the power-law behaviour of the reduced wave functions: the sine and cosine factors do not play any role on the finiteness of the perturbative corrections. Renormalizability requires

$$\frac{5}{2} n_c > \nu + \frac{1}{2}, \quad (58)$$

which in turn implies that the first perturbative correction to the  $C_{2n,l}$  counterterm scales as

$$C_{2n,l} \sim Q^{(5n-1)/2}, \quad (59)$$

that is, the first correction to  $C_0$  enters at order  $Q^{-1/2}$ , the  $C_2$  counterterm at order  $Q^2$ , and so on. It is interesting to notice that (i) the power counting of the counterterms does not depend on the partial wave considered, (ii) the first perturbative correction to  $C_0$  is  $\mathcal{O}(Q^{-1/2})$ , which means that the infrared fixed point is stable, and (iii) the appearance of new counterterms is delayed in comparison with the non-singular potentials.

In contrast, the analysis for repulsive tensor force is puzzling. The decreasing exponential behaviour of the wave function at short distances is able to render the perturbative integral finite, independently of the degree of singularity of the  $\nu$ -th order potential. This feature can be easily appreciated by considering the integral

$$I^{(\nu)} \sim \int_{r_c}^{\infty} \frac{dr}{r^{3+\nu-3/2}} \exp(-4\sqrt{\frac{a_3}{r}}), \quad (60)$$

which is always finite. However, for a consistent power counting to emerge we need new counterterms at higher

orders. Two solutions are possible: (i) follow the power counting rules for attractive singular interaction, that is,  $C_{2n} \sim Q^{(5n-1)/2}$ , (ii) follow naive dimensional analysis, i.e. Weinberg counting, leading to  $C_{2n,l} \sim Q^{2n+2l}$ .

The first solution was proposed by Birse in Ref. [47]. It only considers as relevant to the power counting the power-law behaviour of the wave function, but not the sine, cosine or exponential factor. This expectation is consistent with the fact that the exponentials are just a sine or cosine function with an imaginary argument. However, it is also natural to expect that short distances do not play a significant role if the particles cannot approach each other due to the repulsive potential barrier.

This observation leads to the second solution, namely to follow naive dimensional analysis, which implicitly assumes that the repulsive channel can be treated as a perturbation<sup>7</sup>. For the particular case of the  $^3P_1$  channel, this is not an unreasonable prospect in view of the analysis of perturbative OPE by Birse [47], which suggests a critical momentum around 400 MeV above which the perturbative treatment will fail. In contrast, the critical momentum for the attractive  $^3P_0$  channel is around 200 MeV, requiring the non-perturbative treatment of the OPE tensor force if we want to describe the phase shifts up to  $k \sim 2m_\pi$ . In addition, the critical momenta quoted above were obtained in the  $m_\pi \rightarrow 0$  limit. For finite pion masses we expect the critical momenta to rise, as the  $1/r^3$  behaviour is strongly suppressed for  $m_\pi r > 1$ . In the repulsive case, the potential barrier prevents the two nucleons to explore in detail the  $m_\pi r \sim 1$  region, where the singular nature of the intermediate range tensor force manifests, meaning that the critical momenta are expected to rise significantly with respect to the  $m_\pi \rightarrow 0$  limit. On the contrary, in the attractive case, the two nucleons are driven to the  $m_\pi r \sim 1$  region, and thus the critical momentum will only rise weakly.

However, there is still the problem of what happens if the repulsive singular interaction is non-perturbative.

<sup>7</sup> Alternatively, we can try to analyze the scaling of the wave function at distances of the order of the breakdown scale of the theory,  $r \simeq 0.5$  fm. If the wave function behaves as  $r^{l+1}$  (instead of the singular  $r^{3/4} \exp(-2\sqrt{a_3/r})$ ), naive dimensional expectations are fulfilled and the use of Weinberg counting is justified. Unfortunately, for the particular case of the  $^3P_1$  channel the singular behaviour is achieved at distances of about 1 fm (as  $a_3 = 1.34$  fm in this case), so the wave function scaling argument cannot be applied. The non-perturbative wave function scaling at the breakdown radius does not imply however the failure of the perturbative treatment of OPE. The reason is that the non-perturbative contribution (i.e. from  $r < 1$  fm) to the phase shifts is particularly small, as can be explicitly checked from applying the methods of Ref. [64] to this case, leading to

$$\frac{d\delta_{^3P_1}^{(0)}(k; r_c)}{dr_c} \sim k^3 r_c^{3/2} a_3^{1/2} \exp(-4\sqrt{a_3/r}).$$

The vanishing exponential ensures that, once the non-perturbative regime is achieved, the corresponding contributions to the total phase shift will be negligible.

It is clear that the current understanding of the power counting of repulsive singular potentials is incomplete. Even though in the particular case of the  $^3P_1$  partial wave we can overcome the problem by invoking the perturbative treatment of tensor OPE in this channel, the issue may eventually have important consequences for the power counting of coupled channels. For the moment we will simply assume that attractive and repulsive singular interactions follow the same power counting, unless we expect the singular potential to behave perturbatively. It may be possible that the extension of the methods of Long and Yang [57] from the  $^3P_0$  to the  $^3P_1$  channel will provide new insight into the power counting of repulsive singular interactions.

### C. Coupled Channels

For analyzing the perturbative renormalization of chiral TPE in the coupled channel case, we employ the eigen representation of the phase shifts [71]. In this parametrization the DWBA formulas take their simplest form. The eigen phase shifts are expanded according to the power counting as follows

$$\delta_{\alpha j}(k; r_c) = \sum_{\nu=-1}^{\nu_{\max}} \delta_{\alpha j}^{(\nu)}(k; r_c) + \mathcal{O}(Q^{\nu_{\max}+1}), \quad (61)$$

$$\delta_{\beta j}(k; r_c) = \sum_{\nu=-1}^{\nu_{\max}} \delta_{\beta j}^{(\nu)}(k; r_c) + \mathcal{O}(Q^{\nu_{\max}+1}), \quad (62)$$

$$\epsilon_j(k; r_c) = \sum_{\nu=-1}^{\nu_{\max}} \epsilon_j^{(\nu)}(k; r_c) + \mathcal{O}(Q^{\nu_{\max}+1}), \quad (63)$$

and the LO phase shifts are computed non-perturbatively as explained in Appendix A. The order  $\nu$ -th contribution to the eigen phase shifts is given by

$$\frac{\delta_{\alpha j}^{(\nu)}(k; r_c)}{\sin^2 \delta_{\alpha j}^{(-1)}} = -\frac{2\mu}{k^{2j-1}} \mathcal{A}_{\alpha j}^{(-1)2}(k) I_{\alpha \alpha j}^{(\nu)}(k; r_c), \quad (64)$$

$$\frac{\delta_{\beta j}^{(\nu)}(k; r_c)}{\sin^2 \delta_{\beta j}^{(-1)}} = -\frac{2\mu}{k^{2j+3}} \mathcal{A}_{\beta j}^{(-1)2}(k) I_{\beta \beta j}^{(\nu)}(k; r_c), \quad (65)$$

$$\epsilon_j^{(\nu)}(k; r_c) = -\frac{2\mu}{k^{2j+1}} \frac{\mathcal{A}_{\beta j}^{(-1)}(k) \mathcal{A}_{\alpha j}^{(-1)}(k)}{\cot \delta_{\beta j}^{(-1)} - \cot \delta_{\alpha j}^{(-1)}} I_{\beta \alpha j}^{(\nu)}(k; r_c), \quad (66)$$

where the perturbative integrals  $I_{\alpha \alpha j}$ ,  $I_{\beta \alpha j}$  and  $I_{\beta \beta j}$  are defined as

$$\begin{aligned} I_{\rho \sigma j}^{(\nu)}(k; r_c) = & \int_{r_c}^{\infty} dr \left[ V_{\text{aa}}^{(\nu)}(r) u_{k,\rho j}^{(-1)}(r) u_{k,\sigma j}^{(-1)}(r) + \right. \\ & V_{\text{ab}}^{(\nu)}(r) \left( u_{k,\rho j}^{(-1)}(r) w_{k,\sigma j}^{(-1)}(r) + \right. \\ & \left. \left. w_{k,\rho j}^{(-1)}(r) u_{k,\sigma j}^{(-1)}(r) \right) + \right. \\ & \left. V_{\text{bb}}^{(\nu)}(r) w_{k,\rho j}^{(-1)}(r) w_{k,\sigma j}^{(-1)}(r) \right], \end{aligned} \quad (67)$$

with  $\rho, \sigma = \alpha, \beta$ ; the subscripts a, b in the potential, which take the values  $a = j - 1$  and  $b = j + 1$ , are used to denote the two angular momentum components of the potential ( $l = j \pm 1$ ). In turn, we employ  $\alpha, \beta$  to label the two asymptotic solutions at large distances, which behave as

$$\begin{aligned} B_{\alpha j}^{(-1)} u_{k, \alpha j}^{(-1)}(r) &\rightarrow \cos \epsilon_j^{(-1)} (\cot \delta_{\alpha j}^{(-1)} \hat{j}_{j-1}(kr) - \hat{y}_{j-1}(kr)), \\ B_{\alpha j}^{(-1)} w_{k, \alpha j}^{(-1)}(r) &\rightarrow \sin \epsilon_j^{(-1)} (\cot \delta_{\alpha j}^{(-1)} \hat{j}_{j+1}(kr) - \hat{y}_{j+1}(kr)), \end{aligned} \quad (68)$$

$$\begin{aligned} B_{\beta j}^{(-1)} u_{k, \beta j}^{(-1)}(r) &\rightarrow -\sin \epsilon_j^{(-1)} (\cot \delta_{\beta j}^{(-1)} \hat{j}_{j-1}(kr) - \hat{y}_{j-1}(kr)), \\ B_{\beta j}^{(-1)} w_{k, \beta j}^{(-1)}(r) &\rightarrow \cos \epsilon_j^{(-1)} (\cot \delta_{\beta j}^{(-1)} \hat{j}_{j+1}(kr) - \hat{y}_{j+1}(kr)), \end{aligned} \quad (69)$$

where  $B_{\alpha(\beta)j}^{(-1)}$  are normalization factors, which we define as  $k^{j-1} B_{\alpha j}^{(-1)} = \mathcal{A}_{\alpha j}^{(-1)}(k)$  and  $k^{j+1} B_{\beta j}^{(-1)} = \mathcal{A}_{\beta j}^{(-1)}(k)$ . Finally, for obtaining the perturbative corrections to the phase shifts in the nuclear bar representation [72], which is the most widely used, we consider their relationship with the eigen phases

$$\bar{\delta}_{1j} + \bar{\delta}_{2j} = \delta_{\alpha j} + \delta_{\beta j}, \quad (70)$$

$$\sin(\bar{\delta}_{1j} - \bar{\delta}_{2j}) = \frac{\tan 2\bar{\epsilon}_j}{\tan 2\epsilon_j}, \quad (71)$$

$$\sin(\delta_{\alpha j} - \delta_{\beta j}) = \frac{\sin 2\bar{\epsilon}_j}{\sin 2\epsilon_j}, \quad (72)$$

and re-expand these relations according to the power counting.

As in the uncoupled case, the renormalizability of the coupled channel phase shifts can be analyzed in terms of the  $k^2$  expansion of the perturbative integrals

$$I_{\rho\sigma j}^{(\nu)}(k; r_c) = \sum_{n=0}^{\infty} I_{2n, \rho\sigma j}^{(\nu)}(r_c) k^{2n}, \quad (73)$$

where only the first few terms in the expansion are divergent. The properties of the  $I_{2n, \rho\sigma j}^{(\nu)}$  terms are in turn related to the  $k^2$  expansion of the reduced wave functions, which reads

$$\begin{pmatrix} u_{k, \rho j}(r) \\ w_{k, \rho j}(r) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} u_{2n, \rho j}(r) \\ w_{2n, \rho j}(r) \end{pmatrix} k^{2n}, \quad (74)$$

where  $u_{2n, \rho j}$  and  $w_{2n, \rho j}$  behave as

$$\begin{pmatrix} u_{2n, \rho j}(r) \\ w_{2n, \rho j}(r) \end{pmatrix} \sim \begin{pmatrix} r^{s_{a\rho} + nt_{a\rho}} \\ r^{s_{b\rho} + nt_{b\rho}} \end{pmatrix}, \quad (75)$$

at short enough distances. The specific values of  $s_{l\rho}$  and  $t_{l\rho}$  depend on the details of the LO potential. Owing

to  $t_{l\rho} \geq 2$ , short distances are progressively more suppressed, which in turn implies that the  $I_{2n, \rho\sigma j}^{(\nu)}$  integrals will eventually converge for large enough  $n$ .

From the energy expansion of the reduced wave functions, Eqs. (74) and (75), we can see that the exact degree of divergence of the perturbative integral depends on the specific details of the channel and the phase shift under consideration. However, two simplifications are applicable for the particular case of the nuclear potential in chiral EFT. The first is that all the components of the subleading order potential diverge as  $1/r^{3+\nu}$ , independently of the particular angular momentum subchannels which are involved. The second is that, for an iterated singular interaction at LO, the power law behaviour of the wave function at short distances does not depend on the angular momentum or the  $\alpha$  or  $\beta$  nature of the scattering state: in fact we have  $s = s_{a\alpha} = s_{b\alpha} = s_{a\beta} = s_{b\beta}$  and  $t = t_{a\alpha} = t_{b\alpha} = t_{a\beta} = t_{b\beta}$ . In this case, the situation is analogous to the uncoupled case and the behaviour of the perturbative integrals is given by

$$I_{2n, \rho\sigma}^{(\nu)}(r_c) \sim \int_{r_c}^{\infty} \frac{dr}{r^{3+\nu-2s-nt}}, \quad (76)$$

which diverges for  $3 + \nu - 2s - nt \leq 1$ .

Independently of the previous simplifications, the essential point is that the perturbative integrals can be renormalized by the addition of a certain number of counterterms. As in the uncoupled channel case, the most explicit regularization method is to include  $n_{c(\rho\sigma)}$  free parameters, that is

$$\hat{I}_{\rho\sigma j}^{(\nu)}(k; r_c) = \sum_{n=0}^{n_{c(\rho\sigma)}-1} \lambda_{\rho\sigma}^{(\nu)} k^{2n} + I_{\rho\sigma j}^{(\nu)}(k; r_c), \quad (77)$$

where  $n_{c(\rho\sigma)}$  is the number of counterterms needed to regularize the  $\rho\sigma$  integral. The total number of counterterms is simply  $n_c = n_{c(\alpha\alpha)} + n_{c(\alpha\beta)} + n_{c(\beta\beta)}$ .

The relation between the  $\lambda_{\rho\sigma}^{(\nu)}$  parameters and the counterterms can be obtained by postulating an explicit representation of the short range physics. A convenient representation is

$$\begin{aligned} V_{C, l l'}^{(\nu)}(r; r_c) &= i^{l-l'} \frac{f_l(r_c) f_{l'}(r_c)}{4\pi r_c^2} \\ &\times \sum_{n=0}^{n_{c(l l')} - 1} C_{2n, l l'}^{(\nu)}(r_c) k^{2n} \delta(r - r_c), \end{aligned} \quad (78)$$

with  $f_l(r_c) = \frac{(2l+1)!!}{r_c^l}$  and where  $n_{c(l l')}$  is the number of counterterms between the partial waves  $l$  and  $l'$ . The relation between  $n_{c(l l')}$  and  $n_{c(\rho\sigma)}$  will be discussed below. The momentum space representation of the the previous potential in the  $r_c \rightarrow 0$  limit takes the form

$$\langle p | V_{C, l l'}^{(\nu)} | p' \rangle \rightarrow p^l p'^{l'} \sum_{n=0}^{n_{c(l l')} - 1} C_{2n, l l'}^{(\nu)}(r_c) k^{2n}, \quad (79)$$

that is, they correspond with the usual representation of energy-dependent counterterms. If we have employed the short range normalization of Eq. (32), we obtain the relationship

$$\begin{aligned} \lambda_{2n,\sigma\rho j}^{(\nu)} = & \frac{f_a f_a}{4\pi r_c^2} C_{2n,aa}^{(\nu)}(r_c) u_{\rho,0}(r_c) u_{\sigma,0}(r_c) - \\ & \frac{f_a f_b}{4\pi r_c^2} C_{2n,ab}^{(\nu)}(r_c) \left( u_{\rho,0}(r_c) w_{\sigma,0}(r_c) + \right. \\ & \left. w_{\rho,0}(r_c) u_{\sigma,0}(r_c) \right) + \\ & \frac{f_b f_b}{4\pi r_c^2} C_{2n,bb}^{(\nu)}(r_c) w_{\rho,0}(r_c) w_{\sigma,0}(r_c). \end{aligned} \quad (80)$$

As can be seen, the previous representation of the short range physics requires  $n_{c(aa)} = n_{c(\alpha\alpha)}$ ,  $n_{c(ab)} = n_{c(\alpha\beta)}$  and  $n_{c(bb)} = n_{c(\beta\beta)}$ . This is however not a universal feature: most regulators only require that the total number of counterterms remains the same, and in general we should expect more counterterms in the lower partial waves than with the delta-shell regularization.

A simplification that occurs in the coupled channel case is that the LO interaction is always singular, as a consequence of the tensor force being responsible of coupling channels with different angular momenta. Moreover the singularity structure of the tensor force implies the existence of an attractive and a repulsive subchannel in all the coupled waves, that is, we do not need to discuss the two cases separately. The regular potential case will not be discussed either, but the results are the expected: if we consider the regular solutions, we end up with the standard Weinberg counting. On the contrary, in the particular case of the  ${}^3S_1 - {}^3D_1$  partial wave, if we consider the irregular solution for the s-wave, the KSW counting appears.

### 1. Singular Potential

In coupled triplet channels, the tensor force piece of the OPE potential behaves at short distances ( $r \rightarrow 0$ ) as

$$2\mu \mathbf{V}_{\text{tensor}}^{(-1)}(r) \rightarrow \pm \mathbf{S}_j \frac{a_3}{r^3}, \quad (81)$$

where  $\mathbf{V}$  is a convenient matrix notation for the coupled channel potential, that is

$$(\mathbf{V}^{(-1)})_{ll'} = V_{ll'}^{(\nu)}, \quad (82)$$

with  $l, l' = a, b = j \pm 1$ . The matrix elements of the tensor operator,  $\mathbf{S}^j$ , reads

$$\mathbf{S}_j = \frac{1}{2j+1} \begin{pmatrix} -2(j-1) & 6\sqrt{j(j-1)} \\ 6\sqrt{j(j-1)} & -2(j+2) \end{pmatrix}. \quad (83)$$

Finally,  $a_3$  is a length scale related to the strength of the tensor force.

The interesting point for the regularization of the tensor force is that, at short distances, it can be diagonalized by means of the transformation

$$2\mu \mathbf{R}_j \mathbf{V}_{\text{tensor}}^{(-1)} \mathbf{R}_j^T = \pm \frac{a_3}{r^3} \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}, \quad (84)$$

which implies the existence of a repulsive and attractive eigenchannel. Due to the  $1/r^3$  singularity, the tensor force overcomes the centrifugal barrier at short distances, and the reduced wave functions can be expressed as a sum of the attractive and repulsive solutions. In terms of the  $k^2$  expansion of the reduced wave function, Eq. (74), the following behaviour is expected

$$\begin{pmatrix} u_{2n,\rho j}(r) \\ w_{2n,\rho j}(r) \end{pmatrix} = r^{3/2+5n/2} \begin{pmatrix} g_{a\rho j}(x_A, x_R) \\ g_{b\rho j}(x_A, x_R) \end{pmatrix}, \quad (85)$$

where, depending on the sign of the tensor force, we can either have  $x_A = 2\sqrt{2a_3}/r$ ,  $x_R = 2\sqrt{4a_3}/r$  or  $x_A = 2\sqrt{4a_3}/r$ ,  $x_R = 2\sqrt{2a_3}/r$ . The functions  $g_{a\rho j}$  and  $g_{b\rho j}$  follow the general pattern

$$g(x_A, x_R) = C_S \sin x_A + C_C \cos x_A + C_R e^{-x_R}, \quad (86)$$

where we are only taking the regular solution for the repulsive eigenchannel. The previous behaviour in turn implies that the repulsive solution plays no significant role in the renormalization of the perturbative phase shifts, as it only represents a negligible contribution to the LO wave function at short enough distances.

From the previous form of the reduced wave functions, we have indeed the simplifications which we expected as a consequence of iterating a singular interaction at LO, namely that

$$s = s_{a\alpha} = s_{b\alpha} = s_{a\beta} = s_{b\beta} = 3/4, \quad (87)$$

$$t = t_{a\alpha} = t_{b\alpha} = t_{a\beta} = t_{b\beta} = 5/2. \quad (88)$$

If we compare with the s-wave regular solution of a regular potential ( $s = 1$ ,  $t = 2$ ), the values above imply that the first perturbative correction to  $C_0$  appears half an order earlier than expected, while the  $C_{2n}$ 's happen  $(n-1)/2$  orders later. The finiteness condition reads

$$\frac{5}{2} n_{c(\rho\sigma)} > \nu + \frac{1}{2}, \quad (89)$$

which is independent of whether  $\rho\sigma = \alpha\alpha$ ,  $\alpha\beta$  or  $\beta\beta$ . In terms of the scaling of the counterterms, the previous translates into

$$C_{2n,ll'} \sim Q^{(5n-1)/2}, \quad (90)$$

that is, we obtain the expected scaling resulting from the iteration of a  $1/r^3$  potential. However, due to the existence of the three different phase shifts, the number of counterterms triple with respect to the uncoupled channel case. That is, at order  $Q^{-1/2}$  we have a total of three counterterms, at order  $Q^2$  we reach six counterterms, and so on.

Even though the previous six counterterms guarantee the existence of the  $r_c \rightarrow 0$  limit of the NLO and N<sup>2</sup>LO scattering amplitudes, the existence of the repulsive eigenchannel in tensor OPE means that finiteness may be probably assured with a smaller number of contact operators<sup>8</sup>. The development of this argument requires to work in the basis for which the OPE tensor force is diagonal (i.e. the attractive-repulsive basis), instead of the usual partial wave basis. This is non-trivial from the technical side and the problem remains of which is the adequate power counting for the repulsive eigenchannel. However, if we simply accept the proposal that the repulsive components of the LO tensor force follow the same power counting as the attractive ones [47], the argument of finiteness becomes inconsequential from the power counting point of view and we end up with the aforementioned six counterterms at NLO/N<sup>2</sup>LO. This is the choice we are following for the coupled channels in this work.

Alternatively, we can overcome the limitations in the understanding of the power counting of repulsive singular interactions if we suspect tensor OPE to be perturbative, as happened in the  $^3P_1$  partial wave, in which case we can use the Weinberg counting. There are two possibilities along this line: to work (i) in the attractive-repulsive basis or (ii) in the partial wave basis. The first case is interesting and leads to an exotic power counting<sup>9</sup>, but is also difficult to implement, as already commented in the previous paragraph, and will not be considered in the present work. The second case is much more straightforward and physically compelling and will be analyzed in greater detail in Sect. IV C.

#### D. Convergence of the Perturbative Series

The convergence of the perturbative series can be understood in two different senses, namely with respect to the power counting expansion and with respect to the cut-off. The first kind of convergence is related with the EFT expansion parameter, while the second is linked to the interpretation of the cut-off within the EFT framework.

We do not provide here definitive arguments, but rather educated guesses about these aspects of the theory. In particular, we can establish some bounds on the

breakdown scale and the expansion parameter by considering the related EFT expansion of the chiral potential in the first place. From the power series of the potential, see Eq. (19), it is apparent that the expansion parameter is formally  $x_0 = Q/\Lambda_0$  (independently of which is the concrete value of  $\Lambda_0$ ). Naively, we should also expect a power series in terms of  $x_0$  for the scattering amplitude. However, we can argue on general grounds that the expansion parameter of the amplitude will be bigger than that of the potential. That is, we have

$$T = \sum_{\nu=-1}^{\nu_{\max}} T^{(\nu)} + \mathcal{O}\left(\left(\frac{Q}{\Lambda_1}\right)^{\nu_{\max}+1}\right), \quad (91)$$

instead of Eq. (20), where the real expansion parameter is not  $x_0 = Q/\Lambda_0$ , but rather  $x_1 = Q/\Lambda_1$ , with  $\Lambda_1 (< \Lambda_0)$  the true breakdown scale corresponding to the particular power counting under consideration. The departure from the naive expectation  $\Lambda_1 = \Lambda_0$  is explained in terms of the iteration of the subleading pieces of the chiral potential, which can spoil the convergence of the perturbative expansion of the scattering amplitude.

The previous idea can be illustrated with the KSW counting [44, 45]: in the singlet channel, the KSW breakdown scale is determined by considering the effect of the iteration of OPE on the running of the counterterms [45], yielding  $\Lambda_{1,s} = \Lambda_{NN} \simeq 300$  MeV. This value is much lower than the expected  $\Lambda_0 \sim 0.5$  GeV. The expansion is even less converging in the triplet channel: the analysis of perturbative OPE made by Birse [47] indicates a breakdown scale  $\Lambda_{1,t} \simeq 100$  MeV, in agreement with the results of Ref. [69]<sup>10</sup>. In this particular case we have a clear example in which the convergence is severely limited as a consequence of higher order perturbation theory. That is, the correct identification of the non-perturbative contributions to the interaction is an essential ingredient for a convergent EFT formulation.

In the case of interest for this work, the Nogga et al. counting [21], the breakdown scale can be estimated by the deconstruction method of Refs. [58–60], in which the form of the short range interaction is determined by removing the pion exchange effects from the phenomenological phase shifts. This analysis suggests a breakdown scale of  $\Lambda_{1,s} \simeq 270$  MeV for the  $^1S_0$  and  $^1P_1$  singlet channels [59, 60]. In the triplet channels, the deconstruction has only been performed in the uncoupled p- and d-wave triplets ( $^3P_0$ ,  $^3P_1$  and  $^3D_2$ ), yielding  $\Lambda_{1,t} \simeq 340$  MeV [58]. From the previous results, we expect the approximate expansion parameters  $x_{1,s} = m_\pi/\Lambda_{1,s} \simeq 0.5$  and  $x_{1,t} = m_\pi/\Lambda_{1,t} \simeq 0.4$  for the singlet and triplet channels respectively<sup>11</sup>.

<sup>8</sup> In particular, the two counterterms renormalizing the attractive eigenchannel may be enough. Of course, a formal proof will require to show that the divergences in the perturbative integrals are correlated.

<sup>9</sup> In particular we have  $C_{2n,AA} \sim Q^{(5n-1)/2}$ ,  $C_{2n,AR} \sim Q^{j-5/4+2n}$  and  $C_{2n,RR} \sim Q^{2(j-1)+2n}$ , where A and R refer to the attractive and repulsive eigenchannel respectively. It should be noted that the wave function in the repulsive eigenchannel is a mixture of the  $u_k$  ( $l = j-1$ ) and  $w_k$  ( $l = j+1$ ) wave functions, and consequently it behaves as  $l = j-1$  in terms of the Weinberg counting.

<sup>10</sup> In this regard, the KSW reformulation of Ref. [54] claims to have solved the convergence problems of the triplet channel.

<sup>11</sup> It should be noticed that the deconstruction estimations for the breakdown scale are computed for a cut-off radius of  $r_c = 0.1$  fm. However, as is explained in the next paragraph, EFT calculations

The second type of convergence, that is, the convergence with respect to the cut-off, can also be stated in terms of the potential instead of the scattering amplitude. In this regard, the appearance of the  $1/r^{3+\nu}$  behaviour in the chiral expansion of the potential indicates that the potential is not convergent at short enough distances. If we consider the explicit expansion of the chiral potential at distances below the pion Compton wavelength ( $m_\pi r < 1$ ), that is

$$V_{\text{NN}}(r) \propto \frac{1}{r^3} \sum_{\nu=0}^{\infty} \frac{1}{(\Lambda_0 r)^\nu}, \quad (92)$$

this feature becomes apparent: the Taylor expansion does not converge if  $r < R_0 \sim 1/\Lambda_0$ . Within the framework of a non-perturbative treatment of the chiral potential, this argument requires the coordinate space cut-off to be bigger than the breakdown radius,  $r_c > R_0$ , which we naively expect to lie in the  $R_0 = 0.1 - 0.5$  fm region. Otherwise the scattering amplitudes will start to diverge at high enough orders, independently of the number of counterterms included in the computations. This in turn sets a limit on the value of the cut-off in perturbative calculations. Again, as a consequence of subleading order iterations, the bound of the cut-off will become softer,  $R_1 > R_0$ .

From this observation, a natural interpretation of the cut-off arises within the context of approximation theory, as originally advocated by Stevenson [73]. While the full EFT scattering amplitude (computed at infinite order) is independent of the cut-off in the regions where the potential converges as a consequence of containing an infinite number of counterterms, the truncated EFT scattering amplitude (computed at finite order) is cut-off dependent. However, as far as we have included the necessary counterterms guaranteeing renormalizability<sup>12</sup>, we are free to choose whatever value of the cut-off that provides the better convergence properties towards the full result and realizes the particular power counting under consideration. In this sense, the cut-off is just a parameter controlling the convergence of the theory, as recently stated by Beane et al. [54].

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will not converge below a certain value of the coordinate space cut-off probably around 0.5 fm. Although results computed below that point are not to be trusted, the truth is that the deconstruction estimations are stable up to  $r_c = 0.8$  fm, which is within the values of the cut-off which we regard as acceptable in this work. In this regard, we expect the final value of the expansion parameter to be similar to the estimations based on Refs. [58–60].

<sup>12</sup> This condition is required for the difference between the full result and the calculation at  $\nu$ -th order to formally scale as  $Q^{\nu+1}$ , see also the related comments at the end of Sect. II C.

## E. Power Counting

The power counting arising from the perturbative treatment of chiral TPE is summarized in Table I. The total number of counterterms in the Nogga et al. counting at NLO and N<sup>2</sup>LO ( $n_c = 21$ ) is certainly bigger than the corresponding one in the dimensional Weinberg counting ( $n_c = 9$ ). The counterterm pattern is fundamentally very similar to the power counting obtained by Birse using RGA [47]. However, there are two minor differences with respect to the results of Ref. [47] that are worth commenting: (i) the counting of the  $^3S_1$  partial wave, and (ii) the size of the  $C_{2n}$  operators.

In the RGA of Ref. [47], three counterterms (instead of two) are predicted for the  $^3S_1$  channel. As explained in Ref. [55], this discrepancy arises as a consequence of the naive extension of the idea of trivial and non-trivial fixed points of the RG equations [66, 67], which is only valid for regular long-range potentials, to the singular interaction case in Ref. [47]. However, from the point of view of perturbative renormalizability, the different fixed points stem from the power-law behaviour of the LO wave functions at short distances. In this regard, regular long-range potentials require the existence of two fixed points as a consequence of the existence of a regular and irregular solution. That is, if we have a regular solution the matrix element of the counterterm is

$$\langle V_C \rangle_{\text{reg}} = C(r_c) \int dr \frac{u_k^2(r)}{4\pi r^2} \delta(r - r_c) \sim C(r_c) r_c^{2l}, \quad (93)$$

which requires  $C(r_c)$  to scale as  $r_c^{-2l}$  (i.e.  $Q^{2l}$ ) for having a non-trivial effect. On the other hand, for the irregular solution we have instead

$$\langle V_C \rangle_{\text{irr}} \sim C(r_c) r_c^{-2l-2}, \quad (94)$$

explaining the  $Q^{-2l-2}$  scaling. However, in the particular case of attractive singular potentials, all the short distance solutions of the wave function show the same power law behaviour, and therefore there is only one type of power counting in this case, which is given by the  $Q^{-1/2}$  scaling as can be trivially checked by the previous procedure.

The second difference lies in the order of the energy-dependent operators,  $C_{2n}$ , which is higher in the present formulation ( $C_{2n} \sim Q^{(5n-1)/2}$ ) than in the RGA of Ref. [47] ( $C_{2n} \sim Q^{(4n-1)/2}$ ). A possible explanation is that the  $k^2$  expansion of the reduced wave function for the attractive triplets generates a stronger power law short range suppression than the corresponding one for a regular potential,  $r^{5/2}$  for each power of  $k^2$  instead of  $r^2$ , shifting the  $C_{2n}$  operators to higher orders by a factor of  $Q^{n/2}$ . Taking into account the previous effect, which may have been overlooked in the analysis of Ref. [47] as a result of the presence of two cut-offs, the  $C_{2n}$  operators are demoted from  $Q^{(4n-1)/2}$  to  $Q^{(5n-1)/2}$ , reconciling RGA and perturbative renormalizability. However, the reconciliation is also possible if we consider energy-dependent

contributions to the chiral potential, that is

$$V^{(\nu)}(r) \sim \frac{k^{2m}}{r^{3+\nu-2m}}, \quad (95)$$

where  $2m \leq \nu$ ,  $k = \sqrt{\mu E_{\text{lab}}}$  is the center-of-mass momentum and  $E_{\text{lab}}$  the laboratory energy. This contributions are not forbidden by power counting, but do not occur in the present formulation which employ energy-independent potentials<sup>13</sup>. If they are taken into account, the counterterms should be promoted from  $Q^{(5n-1)/2}$  to  $Q^{(4n-1)/2}$  in the perturbative framework, recovering the initial result from Ref. [47]. Finally, a third explanation is provided by the possibility that the  $r^{5/2}$  suppression related to each subtraction may be contaminated by the low energy scale  $\Lambda_{\text{OPE}}$ . In such a case, the missing scale would imply the reinterpretation of  $r^{5/2}$  as  $r^2 \sqrt{r} \Lambda_{\text{OPE}}$ , thus enhancing the  $C_{2n}$  counterterm by  $Q^{-n/2}$ .

In principle, the discussion is merely academical as it only affects the counting of counterterms beyond the order considered in the present work. However, at N<sup>3</sup>LO ( $Q^4$ ) taking one option or the other can substantially change the counting: if we assume the scaling  $C_{2n} \sim Q^{(5n-1)/2}$ , the third counterterm in triplet channels does not appear until order  $Q^{9/2}$ , between N<sup>3</sup>LO and N<sup>4</sup>LO. On the contrary, if we follow Ref. [47], the triplet channel  $C_4$  counterterms will enter at  $Q^4$ , increasing the total number of free parameters from 27 to 35 at N<sup>3</sup>LO. As the energy-independent representation of the finite range potentials is the preferred one (and as far as the  $\Lambda_{\text{OPE}}$  contamination hypothesis has not been checked), we advocate for the first option. In this regard, the recent power counting analysis of the  $^3P_0$  channel by Long and Yang [57] also prefers the  $Q^2$  scaling (instead of  $Q^{3/2}$ ) for the  $C_2$  operator of the attractive triplet, in agreement with our observations.

#### IV. THE P- AND D-WAVE PHASE SHIFTS

In this section we compute the nucleon-nucleon p- and d-wave phase shifts within the perturbative framework developed in the previous section, that is, we treat the LO potential non-perturbatively and include the subleading order corrections as perturbations. For the counterterms, we follow the results of Table I. The details of the calculations are described in the following paragraphs.

The exact form of the chiral nucleon-nucleon potential in coordinate space is taken from Ref. [74]. We use the parameters  $g_A = 1.26$ ,  $f_\pi = 92.4$  MeV,  $m_\pi = 138.03$  MeV, and  $d_{18} = -0.97$  GeV<sup>2</sup>. The effect of the Goldberger-Treiman discrepancy ( $d_{18}$ ) in the chiral potential can be effectively taken into account by changing the value of  $g_A$  from 1.26 to 1.29 in the LO piece

of the potential, see Refs. [18, 19] for details. For the chiral couplings we use the values  $c_1 = -0.81$  GeV<sup>-1</sup>,  $c_3 = -3.40$  GeV<sup>-1</sup> and  $c_4 = 3.40$  GeV<sup>-1</sup>, which are compatible with the determination of Ref. [75]. We include the relativistic corrections to OPE at NLO and the recoil corrections to TPE at N<sup>2</sup>LO. Following Ref. [74], the relativistic corrections to OPE are included in an energy-dependent manner, that is

$$V_{\text{OPE}}^{(2)}(r) = -\frac{k^2}{2M_N^2} V_{\text{OPE}}^{(0)}(r), \quad (96)$$

where  $k$  is the center of mass momentum and  $M_N = 938$  MeV is the nucleon mass. It should be noted that the s-wave results of Ref. [55] did not include the relativistic correction to OPE. However, explicit calculations show that the effects of the relativistic corrections are fairly small: if we include the contribution from Eq. (96), the s-wave phase shifts do not change appreciably. The same observation applies if we remove the recoil  $1/M_N$  TPE corrections from the results of Ref. [55]. In fact, if we consider the deconstruction estimation for the breakdown scale of the theory, that is,  $\Lambda_{1,s} \sim 300$  MeV and  $\Lambda_{1,t} \sim 350$  MeV, we have the approximate relationship

$$\frac{Q}{M_N} \sim \left( \frac{Q}{\Lambda_1} \right)^2, \quad (97)$$

supporting the original convention of Weinberg for the  $Q/M_N$  terms [10].

The LO phase shifts are computed non-perturbatively by iterating the OPE potential to all orders and by adding the counterterms required to achieve cut-off independence at this order [21]. This condition requires the inclusion of a counterterm in the  $^1S_0$ ,  $^3S_1 - ^3D_1$ ,  $^3P_0$ ,  $^3P_2 - ^3F_2$  and  $^3D_2$  partial waves at LO. In this work the counterterms are included in an implicit manner by fixing the value of the scattering length in the LO phase shifts for the previous channels. The technical details are explained in Appendix A. Following Ref. [55], we employ the cut-off window  $r_c = 0.6 - 0.9$  fm for regularizing the phase shifts. In addition, we show the results for  $r_c = 0.3$  fm for comparison, which also serve as an informal check of the hard cut-off limit of the scattering amplitudes. However, contrary to what happens in the s-waves [55], the variation of the phase shifts in the previous cut-off window can be hardly interpreted as the error band of the results at a given order. The reason is that the cut-off dependence of the p- and d-wave phase shifts is fairly small, sometimes negligible, for cut-off radii below  $r_c = 1.2$  fm.

The subleading order corrections to the phase shifts are computed by making use of the perturbative formalism developed in the previous section. This formalism requires a total of 21 counterterms at NLO/N<sup>2</sup>LO, distributed among the partial waves according to the results of Table I. These counterterms, or equivalently the free parameters which modify the perturbative integrals, see Eqs. (41) and (77), are determined by fitting the NLO and N<sup>2</sup>LO perturbative phase shifts in

<sup>13</sup> With the exception of the relativistic corrections to one pion exchange, for which the energy-dependent representation is chosen.



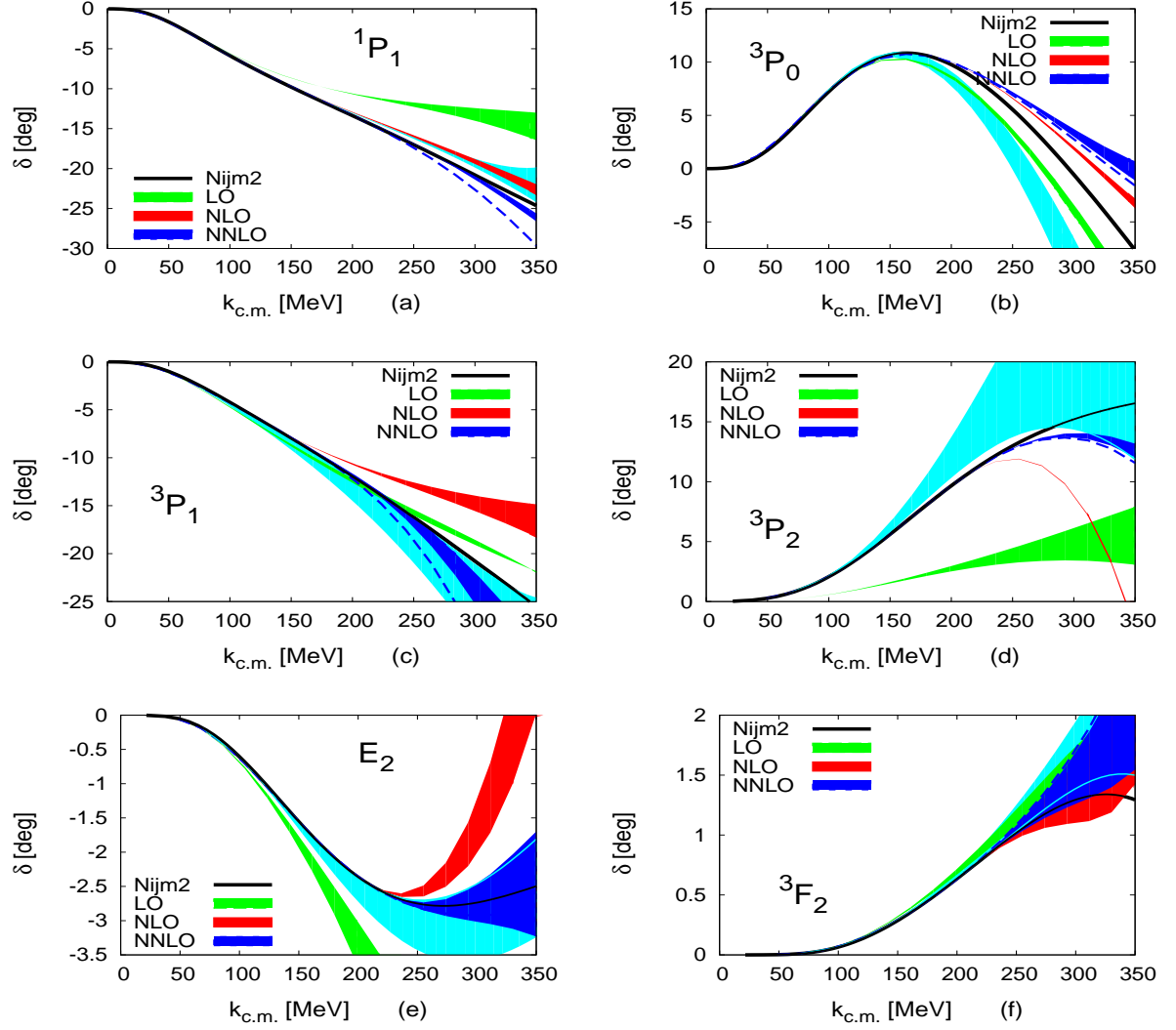


FIG. 3. (Color online) Phase shifts (nuclear bar) for the  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$  and  $^3P_2 - ^3F_2$  channels with the LO piece of the chiral potential (OPE) fully iterated and the NLO and  $N^2$ LO pieces (chiral TPE) treated perturbatively. A LO counterterm is included in the attractive triplets ( $^3P_0$ ,  $^3P_2 - ^3F_2$ ) to remove the uncontrolled cut-off dependence. The contact operators are used to fix the value of the scattering lengths to  $a_{^3P_0} = -2.7 \text{ fm}^3$  and  $a_{^3P_2} = -0.04 \text{ fm}^3$ . The bands are generated by varying the cut-off radius in the region  $r_c = 0.6 - 0.9 \text{ fm}$  and they are formally a higher order effect. However, as a result of the mild cut-off dependence of the results, the full uncertainty of the calculation at a given order is expected to be much higher than the cut-off variation bands. The light blue band corresponds to the  $N^2$ LO results of Ref. [19] in the standard Weinberg counting. The dashed dark blue line represents the  $N^2$ LO results for  $r_c = 0.3 \text{ fm}$ .

the  $k = 100 - 200 \text{ MeV}$  region to the Nijmegen II phase shifts [76], which are in turn equivalent to the Nijmegen PWA [77].

It should be noticed that the perturbative regularization techniques employed in this work are specifically chosen to identify divergences in the amplitudes rather than to optimize the phenomenology. In fact, the regulator we employ can be considered to be the coordinate space equivalent of the sharp regulator in momentum space, which is known to be suboptimal from the phenomenological point of view. Even with this proviso, the phase shifts we obtain are better than the corresponding ones

in the Weinberg scheme at the same order. However, the amplitudes are clearly amenable to improvements.

### A. P-waves

The results for the p-wave phase shifts are shown in Fig. (3). The renormalization of the LO phase shifts requires the inclusion of a counterterm in each of the attractive triplets. Therefore we fix the value of the scattering length in the  $^3P_0$  and  $^3P_2 - ^3F_2$  channels to

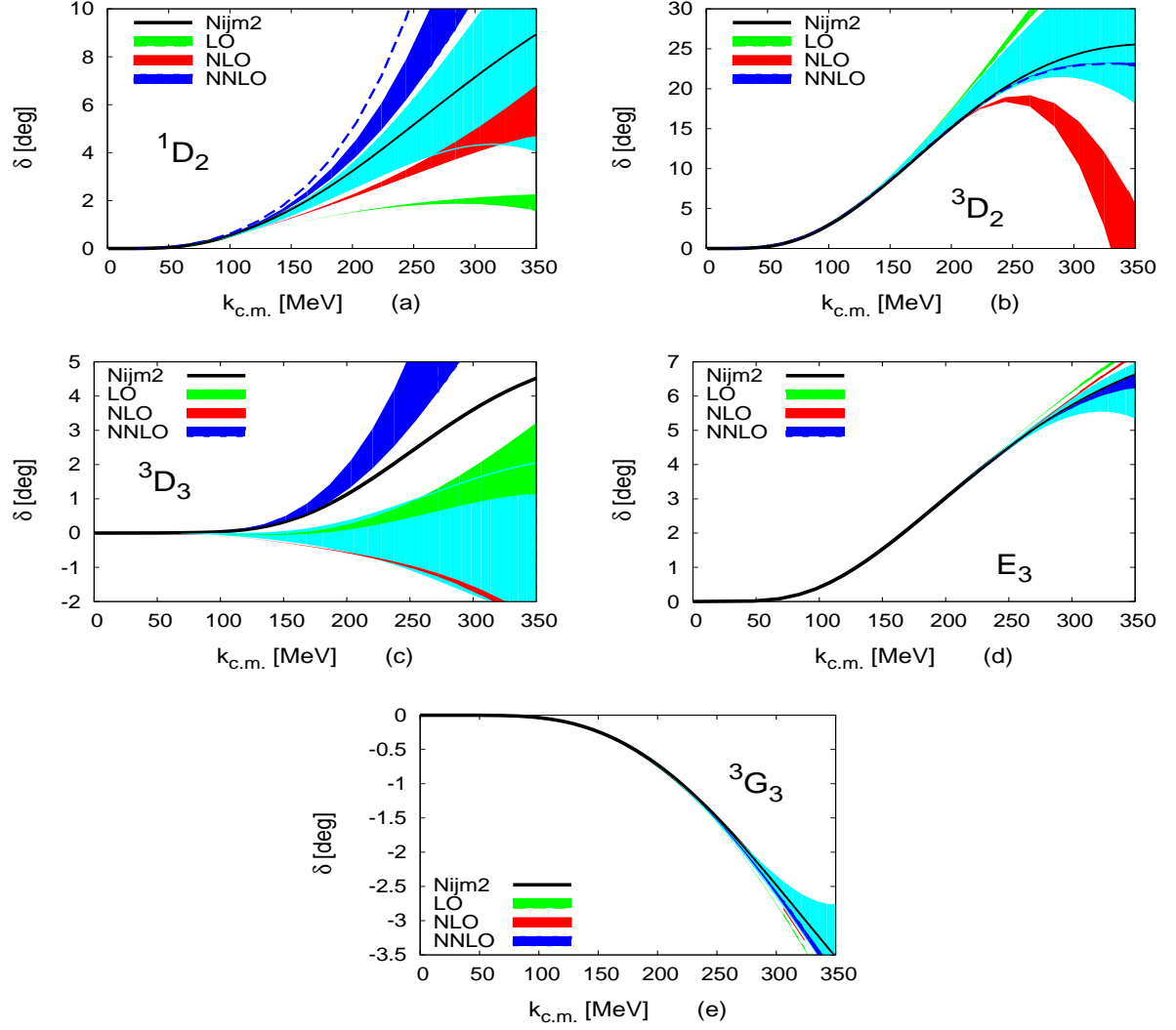


FIG. 4. (Color online) Phase shifts (nuclear bar) for the  $^1D_2$ ,  $^3D_2$  and  $^3D_3 - ^3G_3$  channels, with the LO piece of the chiral potential (OPE) fully iterated and the NLO and N<sup>2</sup>LO pieces (chiral TPE) treated perturbatively. A LO counterterm is included in the  $^3D_2$  channel, as suggested in Ref. [21], which is used to fix the scattering length to the value  $a_{^3D_2} = -7.4 \text{ fm}^5$ . The bands are generated by varying the cut-off radius in the range  $r_c = 0.6 - 0.9 \text{ fm}$  (see the related comments about their interpretation in the previous figure and in the main text). The light blue band corresponds to the N<sup>2</sup>LO results of Ref. [19] in the standard Weinberg counting. The dashed dark blue line represents the N<sup>2</sup>LO results for  $r_c = 0.3 \text{ fm}$ .

$a_{^3P_0} = -2.7 \text{ fm}^3$  and  $a_{^3P_2} = -0.04 \text{ fm}^3$ . The previous values differ from the scattering lengths that can be obtained from the Nijmegen II potential, namely  $a_{^3P_0, \text{Nijm}} = -2.468 \text{ fm}^3$  and  $a_{^3P_2, \text{Nijm}} = -0.2844 \text{ fm}^3$  according to Ref. [78]. The LO deviations of the  $^3P_0$  and  $^3P_2 - ^3F_2$  scattering lengths from the Nijmegen II values are however inconsequential: the LO values should only be accurate up to higher order corrections and there is no need to use the exact value in the LO calculation. The important point for the LO scattering lengths is to provide a good starting point for the power counting expansion. In addition, the mismatch between the peripheral effective range parameters of the Nijmegen poten-

tials and the preferred values of the chiral potential is well known [24, 28, 31, 32].

In general, the description of the p-waves is better than in the Weinberg counting, although this is a natural prospect once we take into account the extra counterterms. In this regard, the  $^1P_1$  and  $^3P_1$  wave are interesting in the sense that they only contain the counterterms prescribed by Weinberg's original counting. Therefore they provide a more direct comparison with the Weinberg scheme, and show that there is no significant drawback to treating TPE perturbatively in these waves. In fact, the results are slightly better than in the standard Weinberg counting.

The phase shifts are nicely reproduced up to around a center-of-mass momentum of  $k_{\text{cm}} = 300$  MeV, although there are signs of convergence up to  $k_{\text{cm}} = 350$  MeV, a figure compatible with deconstruction [58]. Contrary to the s-wave case, the value of the p-wave phase shifts in the chosen cut-off window does not differ much from those obtained when the cut-off is removed. Consequently we expect the deconstruction estimations, which are obtained for  $r_c = 0.1$  fm to work better in the p-waves than in the s-waves.

### B. D-waves

In Fig. (4) we show the results for the d-wave phase shifts. In principle, the complete renormalization of the LO d-wave scattering amplitudes requires the inclusion of a counterterm in the two attractive triplets: the  $^3D_2$  and the  $^3D_3 - ^3G_3$  partial waves. However, we follow here the original suggestion of Ref. [21], in which the counterterm is only (optionally) incorporated in the  $^3D_2$  channel. The reason is that regulator dependence in the  $^3D_3 - ^3G_3$  channel is not apparent until relatively high values of the cut-off [21]. Thus, we can safely obviate the counterterm, at least in the LO calculation. On the other hand, the full iteration of OPE in the  $^3D_3 - ^3G_3$  channel will require the inclusion of six additional counterterms, largely reducing the predictive power of the theory.

In the  $^3D_2$  channel we fix the value of the scattering length to  $a_{^3D_2} = -7.4 \text{ fm}^5$ , which basically coincides with the Nijmegen II value  $a_{^3D_2, \text{Nijm}} = -7.405 \text{ fm}^5$  [78]. In the  $^3D_3 - ^3G_3$  channel, we impose regular boundary conditions for the LO amplitudes. This procedure is of course non-renormalizable, as commented in the previous paragraph, although the LO cut-off dependence does not appear until we reach  $r_c = 0.2$  fm. However, the NLO and N<sup>2</sup>LO cut-off dependence becomes manifest at higher values of the cut-off, around  $r_c = 0.5$  fm in the N<sup>2</sup>LO case. That is, the present treatment of the  $^3D_3 - ^3G_3$  phase shifts is inconsistent. This is also the reason why we do not show the  $r_c = 0.3$  fm results for the  $^3D_3 - ^3G_3$  partial wave in Fig. (4).

If we do not include the LO counterterm, the correct renormalization of the  $^3D_3 - ^3G_3$  channel should require the OPE potential to be of order  $Q^0$  instead of  $Q^{-1}$ . Consequently, the consistent iteration of OPE requires the evaluation of  $(\nu + 1)$ -th order perturbation theory to calculate the  $Q^\nu$  scattering amplitude, a procedure which would generate renormalizable results. On the contrary, the inconsistent full iteration of OPE at LO causes the amplitudes to eventually diverge as

$$\int_{r_c}^{\infty} dr \frac{r^{3/2}}{r^{3+\nu}} \propto \frac{1}{r_c^{1/2+\nu}}, \quad (98)$$

once the LO wave function attains the expected short range behaviour for a solution of a  $1/r^3$  potential. Of course, the implicit assumption behind the present calculation of the  $^3D_3 - ^3G_3$  phase shifts is that the full-

| Partial wave    | LO | NLO | N <sup>2</sup> LO | N <sup>3</sup> LO |
|-----------------|----|-----|-------------------|-------------------|
| $^3P_2 - ^3F_2$ | 1  | 2   | 3                 | 3                 |
| $^3D_3 - ^3G_3$ | 1  | 2   | 2                 | 3                 |
| All             | 6  | 20  | 21                | 26                |

TABLE II. Partial (and total) number of counterterms in the  $^3P_2 - ^3F_2$  and  $^3D_3 - ^3G_3$  partial waves at LO ( $Q^{-1}$ ), NLO ( $Q^2$ ), N<sup>2</sup>LO ( $Q^3$ ) and N<sup>3</sup>LO ( $Q^4$ ) assuming that (i) the iteration of the OPE potential is restricted to the lower partial wave of the coupled channel ( $^3P_2, ^3D_3$ ) and (ii) energy-dependent counterterms. As in the previous case, for the N<sup>3</sup>LO counterterms see the related discussion of Sect. III E.

iterated (or inconsistent) results are not going to be substantially different from the partially-iterated (or consistent) ones, provided the cut-off range is soft enough. For the range of cut-offs we employ ( $r_c = 0.6 - 0.9$  fm) the divergent regime has still not been reached, meaning that the current procedure probably represents a fairly good approximation of a consistent result. In any case, the elucidation of this aspect definitively calls for a perturbative reanalysis of the  $^3D_3 - ^3G_3$  partial wave in order to check this assumption.

It should be commented that the d-wave (and to a lesser extend the f-wave) scattering amplitudes have been traditionally a problem in nuclear EFT, especially at N<sup>2</sup>LO: the excessive attractiveness of the central contribution to the chiral potential at this order generates phase shifts which do not reproduce the results of the partial wave analyses, as first noticed by Kaiser et al. [79, 80]. In particular this poses an issue with respect to the convergence of nuclear EFT in these waves: if the N<sup>2</sup>LO results are still considerably different from the PWA ones, then we should still expect large corrections from higher order contributions and, consequently, a slow convergence rate, rendering the application of nuclear EFT impractical. In this regard, Entem and Machleidt [15] noticed that the inclusion of a counterterm in each of the d-waves in the N<sup>2</sup>LO potential effectively solves the problem, as the short range operators provide the necessary repulsion for compensating the excess of attraction of the long range potential. Curiously, this is not very dissimilar to the situation we encounter within the present power counting scheme. A different solution which is more compatible with the application of the Weinberg counting is the use of spectral function regularization, as advocated by Epelbaum et al. [18, 19]. In spectral function regularization a second cut-off is introduced to regularize the momentum in the pion loops, a procedure which greatly reduces the strength of the N<sup>2</sup>LO potential, effectively solving the convergence problem without the addition of new counterterms. Of course, the most obvious solution is the use of softer cut-offs, as happens for example with the Nijmegen  $\chi$ PWA's [74, 81].

As we can see, the previous problem manifests moderately in the d-wave phase shifts of Fig. (4). In partic-

ular, it only affects the  $^1D_2$  and  $^3D_3$  phases, although in a milder form than in Ref. [79]. In any case, the excess of attraction in the  $^1D_2$  and  $^3D_3$  channels for the  $r_c = 0.6 - 0.9$  fm cut-off window is still under control, and we do not expect to have convergence problems for  $k_{\text{cm}} \leq 300$  MeV at N<sup>3</sup>LO, when the first counterterm is added. However, it is clear that a softer cut-off window will probably improve the convergence rate of these two partial waves without worsening the other p- and d-wave phase shifts. Finally, the  $^3D_2$  partial wave is well-described and shows signs of convergence up to  $k_{\text{cm}} = 350$  MeV, as expected from deconstruction [58].

### C. Improved Power Counting Scheme for the Coupled Channels

A particularly problematic feature of iterating OPE in the coupled triplet channels is the associated requirement of including a total of six counterterms. For the  $^3S_1 - ^3D_1$  channel, this condition is not a significant drawback. However, the counterterm proliferation caused by the full iteration of OPE starts to become worrisome in the  $^3P_2 - ^3F_2$  channel and excessive in the  $^3D_3 - ^3G_3$  partial wave. The reason for the six counterterm figure is that the iteration of OPE in the coupled channels assumes the same short-range behaviour for the two-linearly independent LO wave functions which correspond to the  $\alpha$  and  $\beta$  scattering states. This justifies the necessity of two counterterms per phase: the scaling of counterterms is identical independently of whether they only affect the  $^3P_2$  ( $^3D_3$ ) partial wave, the  $^3F_2$  ( $^3G_3$ ), or the admixture between them. However, it is natural to expect that the  $r^{3/4}$  wave function behaviour will be first achieved in the  $\alpha$  (rather than in the  $\beta$ ) scattering state.

Along the previous lines we analyze here an alternative power counting scheme for the peripheral coupled triplets which (i) reduces the number of counterterms without significantly worsening the phenomenological description of the phase shifts in the  $^3P_2 - ^3F_2$  channel and (ii) limits the counterterm proliferation if we decide to iterate OPE in the  $^3D_3 - ^3G_3$  partial wave. The idea is to only iterate tensor OPE in the  $\alpha$  scattering state of the coupled channel, that is, in the lower partial wave. For the particular case of the  $^3P_2 - ^3F_2$  channel what we mean is that, while we have naively expected the full OPE potential to be promoted to order  $Q^{-1}$ , that is

$$\begin{pmatrix} V_{pp}^{(0)} & V_{pf}^{(0)} \\ V_{fp}^{(0)} & V_{ff}^{(0)} \end{pmatrix} \rightarrow \begin{pmatrix} V_{pp}^{(-1)} & V_{pf}^{(-1)} \\ V_{fp}^{(-1)} & V_{ff}^{(-1)} \end{pmatrix}, \quad (99)$$

it is much better to do the following promotion

$$\begin{pmatrix} V_{pp}^{(0)} & V_{pf}^{(0)} \\ V_{fp}^{(0)} & V_{ff}^{(0)} \end{pmatrix} \rightarrow \begin{pmatrix} V_{pp}^{(-1)} & V_{pf}^{(0)} \\ V_{fp}^{(0)} & V_{ff}^{(0)} \end{pmatrix} \quad (100)$$

in which only the diagonal p-wave piece of the potential is iterated. In this scheme, we have one and two

(p- to p-wave) counterterms at orders  $Q^{-1}$  and  $Q^2$  respectively, while the first counterterm mixing the p- and f-waves enters at order  $Q^{2+3/4}$ . Taking into account the smallness of the  $\bar{\epsilon}_2$  mixing parameter and the  $\bar{\delta}_{3F_2}$  phase shift, the previous modification does not seem an unreasonable prospect. An advantage of this scheme is that the size of the matrix elements of the two-pion exchange potential are reduced in the  $pf$  and  $ff$  channels, thus decreasing the size of the subleading contributions with respect to the naive implementation of the counting of Ref. [21]. For the  $^3D_3 - ^3G_3$  partial wave we propose the following promotion

$$\begin{pmatrix} V_{dd}^{(0)} & V_{dg}^{(0)} \\ V_{gd}^{(0)} & V_{gg}^{(0)} \end{pmatrix} \rightarrow \begin{pmatrix} V_{dd}^{(-1)} & V_{dg}^{(0)} \\ V_{gd}^{(0)} & V_{gg}^{(0)} \end{pmatrix}, \quad (101)$$

which implies one counterterm at LO for the  $dd$  subchannel, and two at NLO and N<sup>2</sup>LO. The first counterterm mixing the d- and g-wave subchannels appears at order  $Q^{3+3/4}$ , that is, between N<sup>2</sup>LO and N<sup>3</sup>LO. The power counting resulting from these modifications is summarized in Table II.

The power counting proposed here can be explained on the basis of a variation of a well-known argument [21, 57] that explains the perturbative character of OPE in peripheral waves. We analyze the behaviour of the partial wave projection of the OPE potential at low momenta, which is given by

$$\langle p | V_{\text{OPE},ll'} | p' \rangle \sim \frac{c_{ll'}}{\mu \Lambda_{\text{OPE}}} \frac{p^l p'^{l'}}{m_\pi^{l+l'}}, \quad (102)$$

where  $\Lambda_{\text{OPE}}$  is the scale governing the strength of OPE potential,  $l$  and  $l'$  are the angular momenta of the partial waves and  $c_{ll'}$  a geometrical factor that approximately behaves as  $1/(l!l'!)$  [21]. If  $\Lambda_{\text{OPE}}$  is considered to be a low energy scale, the order of the OPE potential will be  $Q^{-1}$  (instead of the naive  $Q^0$ ). However, owing to the far from perfect separation of scales, the geometrical factor  $c_{ll'}$  is eventually able to mimic the effect of positive powers of  $Q/\Lambda_0$  (or more properly  $Q/\Lambda_1$ , see Sect.III D for details)<sup>14</sup>. In the nuclear EFT proposed in the present work, we expect a expansion parameter of the order of 0.4 for the triplet, which means that for  $l$  or  $l' \geq l_{\text{crit}}$  (with  $l_{\text{crit}} = 2$  or 3) the angular momentum suppression is equivalent to the demotion of OPE by at least one order. For higher partial waves the actual demotion of OPE will be even higher, proportional to several orders in the chiral expansion, thus explaining the fact that first order perturbation theory for OPE works better the more peripheral the partial wave [79]. For keeping matters simple, we have only considered the demotion of OPE of by at most one order for  $l \geq 3$ .

<sup>14</sup> There exists the possibility that a similar geometrical suppression is taking place for repulsive tensor forces when treated non-perturbatively, reconciling power counting with the naive expectation that short range interactions should be less important in the repulsive case compared to the attractive one.

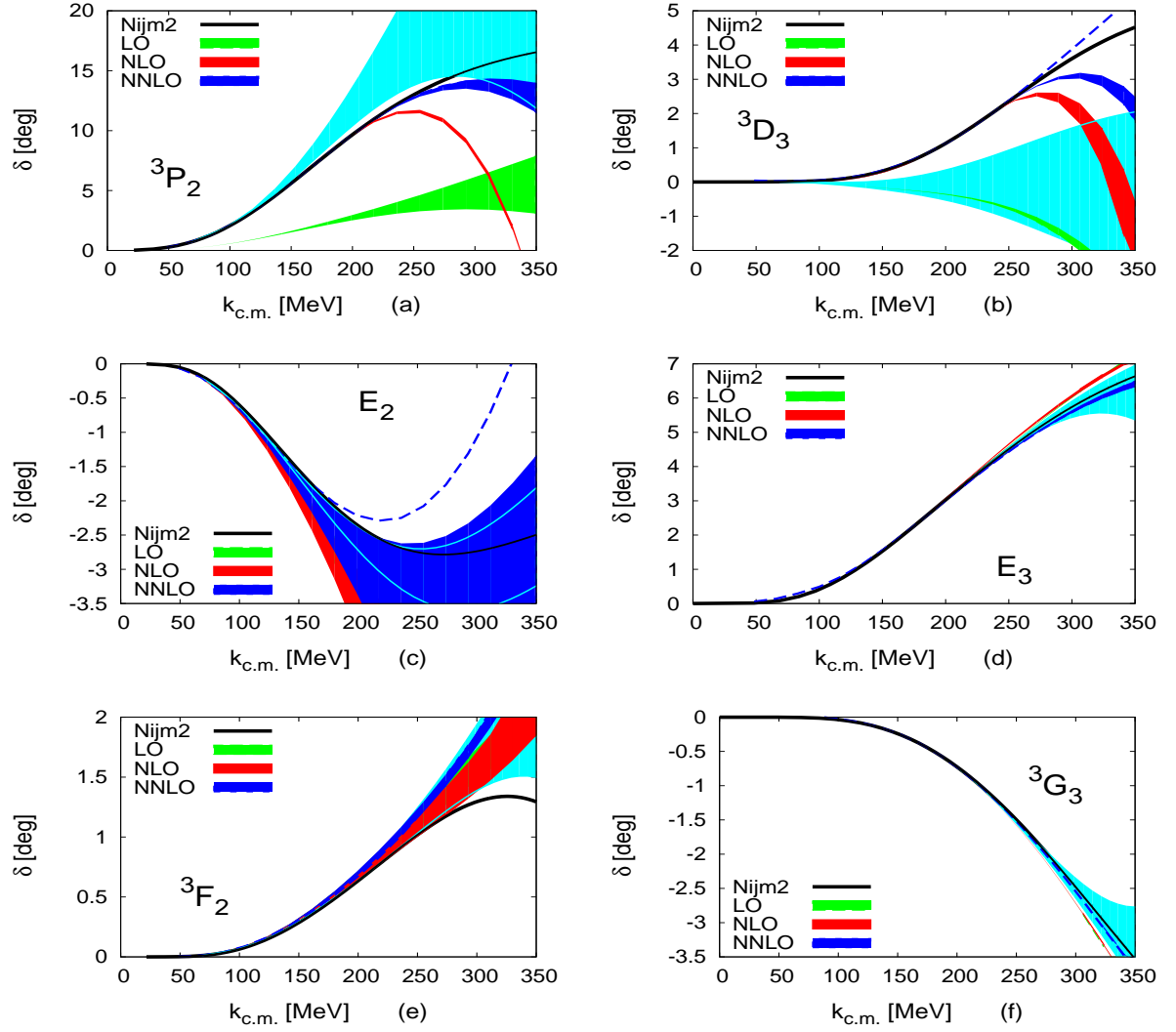


FIG. 5. (Color online) Phase shifts (nuclear bar) for the  ${}^3P_2 - {}^3F_2$  and  ${}^3D_3 - {}^3G_3$  coupled channels in a power counting scheme in which the OPE potential is fully iterated only in the lower partial wave of the coupled channel, while the other contributions are treated as perturbations. The LO counterterm are employed to fix the value of the scattering length to  $a_{3P_2} = -0.04 \text{ fm}^3$  and  $a_{3D_3} = -0.085 \text{ fm}^5$ . The light blue band corresponds to the  $\text{N}^2\text{LO}$  results of Ref. [19] in the standard Weinberg counting. The dashed dark blue line represents the  $\text{N}^2\text{LO}$  results for  $r_c = 0.3 \text{ fm}$ .

Of course, the consistent implementation of the previous power counting scheme requires the evaluation of perturbation theory up to third (fourth) order at NLO ( $\text{N}^2\text{LO}$ ). However, owing to the computational difficulties related to the evaluation of the perturbative series at high orders, we will fully iterate the OPE potential as in previous cases. Although this is inconsistent from the power counting point of view, and probably even non-renormalizable, we expect this simplified calculation to be a good approximation of the true results in this improved counting scheme. The regularization of the subleading order amplitudes is performed by assuming a delta-shell parametrization of the counterterms, see Eq. (78). The corresponding  $\lambda$ -parameters to be added

in the perturbative integrals can be obtained by means of Eq. (80). The counterterms are then fitted to the Nijmegen II phase shifts in the  $k_{\text{cm}} = 100 - 200 \text{ MeV}$  region. The amplitudes generated by this particular regularization procedure seem to be fairly cut-off independent: no sign of divergent behaviour has been found for  $r_c \geq 0.3 \text{ fm}$ . Numerical limitations have prevented the exploration of the harder cut-off region below  $0.3 \text{ fm}$ . Consequently we are unable to determine the eventual renormalizability of the regularization we have employed<sup>15</sup>.

<sup>15</sup> Conversely, the direct inclusion of  $\lambda$ -parameters only in the  $I_3 P_2$

The results are shown in Fig. (5). The LO phase shifts have been regularized by fixing the scattering length to  $a_{^3P_2} = -0.04 \text{ fm}^3$  and  $a_{^3D_3} = -0.085 \text{ fm}^5$ . In the  $^3P_2 - ^3F_2$  channel the improved power counting scheme yields similar results to the original one in Fig. (3), but employing half the number of parameters. The results for the  $^3D_3 - ^3G_3$  have also improved, as expected from the inclusion of two new counterterms. The convergence properties are similar to other attractive triplets, and the  $r_c = 0.3 \text{ fm}$  results do not significantly differ from the  $r_c = 0.6 - 0.9 \text{ fm}$  ones.

## V. DISCUSSION AND CONCLUSIONS

In the present work we have explored in detail the extension of the power counting proposal of Nogga, Timmermans and van Kolck [21] to the subleading orders of the chiral expansion up to  $N^2\text{LO}$  ( $Q^3$ ). We have chosen a perturbative regularization framework which facilitates the identification of divergent contributions and therefore the subsequent identification of the correct power counting for the contact operators. In particular, the present scheme fulfills the demanding constraints of renormalizability and power counting without neglecting the phenomenological aspects: the results are in fact as good as, if not better than the corresponding ones in the Weinberg counting at the same order.

We have determined the power counting of the contact operators, resulting in a total of 21 counterterms for the s-, p- and d-waves at NLO and  $N^2\text{LO}$ , that is, twelve more than in Weinberg dimensional counting at the same order. The distribution of the contact operators largely agrees with the related RGA by Birse [47], but we also include several clarifications and improvements. In the  $^3P_1$  channel we explain the power counting disagreement between RGA [47] and the original exploration by Nogga et al. [21]. We also propose modifications which enhance the consistency of the approach and improve the counterterm distribution in the  $^3P_2 - ^3F_2$  and  $^3D_3 - ^3G_3$  channels. However, a serious assessment of power counting in the previous cases requires a perturbative reanalysis of the OPE potential up to fourth order in the perturbative series.

In principle, the perturbative treatment of the subleading pieces of the chiral interaction raises the problem of a serious departure with respect to the traditional approach to nuclear physics, namely the idea of using a potential as the elementary computational building block in few and many body calculations. However, the divorce is merely apparent. Perturbative renormalization should be understood as an analysis tool of the formal aspects of the theory, rather than as the adequate framework for

final calculations. The point is not to explicitly treat the subleading orders of the chiral two-body forces perturbatively, but the fact that they only represent a small contribution to physical observables.

In this respect, a sensible approach is to construct chiral two- and three-nucleon potentials that can be shown (i) to contain the subleading pieces of the interaction approximately as perturbations, therefore complying with power counting expectations, and (ii) to be cut-off independent modulo higher order uncertainties in a reasonable range of cut-offs. Non-perturbative methods may fulfill these conditions provided an adequate regulator function and cut-off window are employed, as proposed in Ref. [61]. In particular, the perturbative character of chiral TPE can always be checked *a posteriori*. If we consider the pioneering work of Ray, Ordoñez and van Kolck [11, 12] as the first generation of chiral potentials, and the phenomenologically successful potentials of Entem and Machleidt [17] and Epelbaum, Glöckle and Meißner [20] as the second generation, what we need is a third generation of chiral potentials which overcomes the theoretical deficiencies of previous attempts. This task will surely require much less work than the remaking of all of nuclear physics in a perturbative fashion.

The prospects for the feasibility of this approach are indeed good. In this regard, there exists the exciting possibility of reinterpreting the Nijmegen  $\chi\text{PWA99}$  [74] as an EFT calculation in the present power counting scheme. In the chiral PWA of Ref. [74], the authors regularize the chiral  $N^2\text{LO}$  potential with a total of 23 (22) boundary condition at the cut-off radius  $r_c = 1.4$  (1.8) fm, achieving a  $\chi^2/d.o.f. \simeq 1$  in the  $E_{\text{lab}} \leq 350 \text{ MeV}$  region<sup>16</sup>. Taking into account the size of the cut-off radius they use, chiral TPE is almost guaranteed to be a perturbation, especially in view of the perturbative analysis of Ref. [82]. In the power counting advocated in this work, we employ a total of 21 parameters for the s-, p- and d-waves, a figure suspiciously similar to Ref. [74]. However, explicit calculations would be required to check whether the  $\chi\text{PWA99}$  can be considered a realization of the counting proposal of Nogga et al. [21].

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( $I_{^3D_3}$ ) and eventually the  $I_{E_2}$  perturbative integrals generate divergences in all the integrals that do not contain at least two  $\lambda$ -parameters.

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<sup>16</sup> Note that we are not taking into account the more recent  $\chi\text{PWA03}$  of Ref.[81], which claims to describe the nucleon-nucleon scattering data up to  $E_{\text{lab}} = 500 \text{ MeV}$  with a total of 33 boundary conditions at  $r_c = 1.6 \text{ fm}$ . The reason is that energy range of Ref.[81] extends well beyond the expected range of applicability of nuclear EFT.

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## Appendix A: Renormalization of the Leading Order Phase Shifts

In this appendix we explain the non-perturbative renormalization of the LO scattering amplitudes and wave functions. The appendix can be considered a review of the coordinate space renormalization techniques of Refs. [22–24]. We consider first the uncoupled channel case and then move to the coupled channel case.

### 1. Uncoupled Channels

We start by defining the reduced Schrödinger equation for the LO reduced wave function  $\hat{u}_{k,l}^{(-1)}$

$$-\hat{u}_{k,l}^{(-1)''} + 2\mu \left[ V^{(-1)}(r) + \frac{l(l+1)}{r^2} \right] \hat{u}_{k,l}^{(-1)} = k^2 \hat{u}_{k,l}^{(-1)}, \quad (\text{A1})$$

which we consider to be valid from  $r_c$  to  $\infty$ , where  $r_c$  is the cut-off radius. With  $k$  and  $l$  we refer to the center of mass momentum and the orbital angular momentum respectively, and  $V^{(-1)}(r)$  is the LO potential in configuration space. The asymptotic ( $r \rightarrow \infty$ ) normalization of the  $\hat{u}_k$  wave function is given by

$$\hat{u}_{k,l}^{(-1)}(r) \rightarrow k^l \left( \cot \delta_l^{(-1)} \hat{j}_l(kr) - \hat{y}_l(kr) \right), \quad (\text{A2})$$

where  $\delta_l^{(-1)}(k; r_c)$  is the LO phase shifts.

The regularization and calculation of the phase shifts depend on whether we include a counterterm at LO or not. We will only consider in detail the first case, as the second can be handled by standard means. The addition of a counterterm is equivalent to the condition of fixing the scattering length in the zero energy solution. For this, we set the asymptotic form of the zero energy solution to

$$\hat{u}_{0,l}^{(-1)}(r) \rightarrow \frac{(2l-1)!!}{r^l} - \frac{r^{l+1}}{(2l+1)!!} \frac{1}{a_l^{(-1)}}, \quad (\text{A3})$$

where  $a_l^{(-1)}$  is the LO value of the scattering length, which we fix to the desired value. We integrate the zero energy solution downwards, from  $\infty$  to  $r_c$  in Eq. (A1). Then, we assume the logarithmic derivatives of the zero and finite energy solutions to be equal at  $r = r_c$

$$\frac{\hat{u}_{k,l}^{(-1)'}(r_c)}{\hat{u}_{k,l}^{(-1)}(r_c)} = \frac{\hat{u}_{0,l}^{(-1)'}(r_c)}{\hat{u}_{0,l}^{(-1)}(r_c)}, \quad (\text{A4})$$

which effectively provides the initial integration conditions for obtaining the finite energy solution at arbitrary radius  $r$ . Finally we integrate upwards and obtain the

phase shifts by comparing the wave function at large radii with the asymptotic form of Eq. (A2).

However, for the previous wave functions to be useful in determining the divergence of the perturbative integrals we need to change the normalization. In the normalization defined by Eq. (A2), the reduced wave function  $\hat{u}_{k,l}^{(-1)}$  is not necessarily energy independent at the cut-off radius, making more difficult the identification of the required counterterms. We therefore define a new normalization of the wave function which is energy independent at  $r = r_c$

$$\hat{u}_{k,l}(r) = \mathcal{A}_l^{(-1)}(k; r_c) u_{k,l}(r). \quad (\text{A5})$$

The normalization factor  $\mathcal{A}_l^{(-1)}(k; r_c)$  can be uniquely determined by requiring the additional condition  $\mathcal{A}_l^{(-1)}(0; r_c) = 1$ , in which case we have

$$\mathcal{A}_l^{(-1)}(k; r_c) = \frac{\hat{u}_{k,l}(r_c)}{\hat{u}_{0,l}(r_c)}. \quad (\text{A6})$$

### 2. Coupled Channels

For the coupled channel case the reduced Schrödinger equation reads

$$-\hat{u}_{k,j}^{(-1)''} + 2\mu \left[ V_{aa}^{(-1)}(r) + \frac{(j-1)j}{r^2} \right] \hat{u}_{k,j}^{(-1)} + 2\mu V_{ab}^{(-1)}(r) \hat{w}_{k,j}^{(-1)} = k^2 \hat{u}_{k,j}^{(-1)}, \quad (\text{A7})$$

$$-\hat{w}_{k,j}^{(-1)''} + 2\mu \left[ V_{bb}^{(-1)}(r) + \frac{(j+1)(j+2)}{r^2} \right] \hat{w}_{k,j}^{(-1)} + 2\mu V_{ba}^{(-1)}(r) \hat{u}_{k,j}^{(-1)} = k^2 \hat{w}_{k,j}^{(-1)}, \quad (\text{A8})$$

where  $j$  is the total angular momentum and  $\hat{u}$  ( $\hat{w}$ ) the  $l = j - 1$  ( $l = j + 1$ ) wave function. For the LO potential we employ the  $a$  ( $b$ ) subscripts to denote the  $l = j - 1$  ( $l = j + 1$ ) subchannel in the potential matrix. Of course, we assume the Schrödinger equation to be valid only from  $r = r_c$  to  $\infty$ .

For the coupled channel case we can define two linearly independent solutions for  $r \rightarrow \infty$ , which are the  $\alpha$  and  $\beta$  scattering states. In the Blatt-Biedenharn parametrization of the phase shifts [71] their asymptotic behaviour

is given by

$$\hat{u}_{k,\alpha j}^{(-1)}(r) \rightarrow k^{j-1} \cos \epsilon_j^{(-1)} \times (\cot \delta_{\alpha j}^{(-1)} \hat{j}_{j-1}(kr) - \hat{y}_{j-1}(kr)), \quad (\text{A9})$$

$$\hat{w}_{k,\alpha j}^{(-1)}(r) \rightarrow k^{j-1} \sin \epsilon_j^{(-1)} \times (\cot \delta_{\alpha j}^{(-1)} \hat{j}_{j+1}(kr) - \hat{y}_{j+1}(kr)), \quad (\text{A10})$$

$$\hat{u}_{k,\beta j}^{(-1)}(r) \rightarrow -k^{j+1} \sin \epsilon_j^{(-1)} \times (\cot \delta_{\beta j}^{(-1)} \hat{j}_{j-1}(kr) - \hat{y}_{j-1}(kr)), \quad (\text{A11})$$

$$\hat{w}_{k,\beta j}^{(-1)}(r) \rightarrow k^{j+1} \cos \epsilon_j^{(-1)} \times (\cot \delta_{\beta j}^{(-1)} \hat{j}_{j+1}(kr) - \hat{y}_{j+1}(kr)). \quad (\text{A12})$$

The renormalization of a singular potential in coupled channels depends on the number of attractive or repulsive eigenchannels at short distances [23, 24]. Here we will only consider in detail the kind of regularization which appears when the LO interaction is the OPE potential [22]. In such a case, we only need to fix one parameter to renormalize the phase shifts. This parameter is usually selected to be the scattering length associated with the  $\alpha$  phase shift,  $\delta_{\alpha j}$ , but in principle there is no impediment to fix any other low energy observable. For fixing a scattering length we take into account that the wave function of the zero energy  $\alpha$  and  $\beta$  states behave asymptotically as

$$\hat{u}_{0,\alpha j}^{(-1)}(r) \rightarrow \frac{(2j-3)!!}{r^{j-1}} - \frac{r^j}{(2j-1)!!} \frac{1}{a_{\alpha j}}, \quad (\text{A13})$$

$$\hat{w}_{0,\alpha j}^{(-1)}(r) \rightarrow \frac{(2j+1)!!}{r^{j+1}} e_j, \quad (\text{A14})$$

$$\hat{u}_{0,\beta j}^{(-1)}(r) \rightarrow \frac{r^j}{(2j-1)!!} \frac{e_j}{a_{\beta j}}, \quad (\text{A15})$$

$$\hat{w}_{0,\beta j}^{(-1)}(r) \rightarrow \frac{(2j+1)!!}{r^{j+1}} - \frac{r^{j+2}}{(2j+3)!!} \frac{1}{a_{\beta j}}, \quad (\text{A16})$$

where  $a_{\alpha j}$ ,  $e_j$  and  $a_{\beta j}$  are the scattering lengths related to the  $\delta_{\alpha j}$ ,  $\epsilon_j$  and  $\delta_{\beta j}$  phases (i.e.  $\delta_{\alpha j} \rightarrow -a_{\alpha j}k^{2j-1}$ ,  $\epsilon_j \rightarrow e_j k^2$ ,  $\delta_{\beta j} \rightarrow -a_{\beta j}k^{2j+3}$ ), see Ref. [24] for details. As previously suggested, we fix  $a_{\alpha j}$  to the desired value, leaving  $e_j$  momentarily free, and integrate the zero energy  $\alpha$  state downwards from  $r \rightarrow \infty$  to  $r = r_c$ . At the cut-off radius we will impose boundary conditions which will determine the initial integration conditions for the finite energy  $\alpha$  and  $\beta$  scattering states.

There are several possible regularization conditions for the coupled channels [22]. In the present work, we will employ the following set

$$\hat{u}_{k,\sigma}(r_c) = f_j \hat{w}_{k,\sigma}(r_c), \quad (\text{A17})$$

$$\frac{\hat{u}_{k,\sigma}(r_c)}{f_j \hat{u}'_{k,\sigma}(r_c) + \hat{w}'_{k,\sigma}(r_c)} = \frac{\hat{u}_{0,\rho}(r_c)}{f_j \hat{u}'_{0,\rho}(r_c) + \hat{w}'_{0,\rho}(r_c)}, \quad (\text{A18})$$

where  $\sigma, \rho = \alpha, \beta$  and with  $f_j$  a parameter which is set to  $f_j = -\sqrt{\frac{j}{j+1}}$  for  $j$  even and  $f_j = \sqrt{\frac{j+1}{j}}$  for  $j$  odd.

The advantage of these particular regularization conditions is that they facilitate the energy independent normalization of the wave functions at short distances. We define the new normalization as follows

$$\hat{u}_{k,\sigma j}^{(-1)} = \mathcal{A}_{\sigma j}^{(-1)}(k; r_c) u_{k,\sigma j}^{(-1)}, \quad (\text{A19})$$

$$\hat{w}_{k,\sigma j}^{(-1)} = \mathcal{A}_{\sigma j}^{(-1)}(k; r_c) w_{k,\sigma j}^{(-1)}, \quad (\text{A20})$$

where  $\sigma = \alpha, \beta$ . By requiring  $\mathcal{A}_{\sigma j}^{(-1)}(0; r_c) = 1$ , we obtain the straightforward relationships

$$\mathcal{A}_{\sigma j}^{(-1)}(k; r_c) = \frac{\hat{u}_{k,\sigma j}^{(-1)}(r_c)}{\hat{u}_{0,\sigma j}^{(-1)}(r_c)} = \frac{\hat{w}_{k,\sigma j}^{(-1)}(r_c)}{\hat{w}_{0,\sigma j}^{(-1)}(r_c)}. \quad (\text{A21})$$

As can be seen, the regularization condition given by Eq. (A17) assures that the normalization factor  $\mathcal{A}_{\sigma j}^{(-1)}$  is well defined independently of whether we use the  $\hat{u}$  or  $\hat{w}$  wave function for computing it. Other regularization conditions do not guarantee this property, thus generating a spurious energy dependence of one of the components of the wave function at finite cut-off radii.

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- [1] S. Beane, P. Bedaque, K. Orginos, and M. Savage, Phys.Rev.Lett. **97**, 012001 (2006), arXiv:hep-lat/0602010 [hep-lat]
  - [2] N. Ishii, S. Aoki, and T. Hatsuda, Phys.Rev.Lett. **99**, 022001 (2007), arXiv:nucl-th/0611096 [nucl-th]
  - [3] T. Doi, S. Aoki, T. Hatsuda, Y. Ikeda, T. Inoue, *et al.*(2011), \* Temporary entry \*, arXiv:1106.2276 [hep-lat]
  - [4] S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips, and M. J. Savage, At the Frontiers of Particle Physics: Handbook of QCD, Vol 4, World Scientific Publishing Company(2002)
  - [5] P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. **52**, 339 (2002), nucl-th/0203055

- [6] E. Epelbaum, Prog. Part. Nucl. Phys. **57**, 654 (2006), arXiv:nucl-th/0509032
- [7] E. Epelbaum, H.-W. Hammer, and U.-G. Meißner, Rev. Mod. Phys. **81**, 1773 (2009), arXiv:0811.1338 [nucl-th]
- [8] R. Machleidt and D. Entem, Phys.Rept. **503**, 1 (2011), arXiv:1105.2919 [nucl-th]
- [9] S. Weinberg, Phys. Lett. **B251**, 288 (1990)
- [10] S. Weinberg, Nucl. Phys. **B363**, 3 (1991)
- [11] C. Ordoñez, L. Ray, and U. van Kolck, Phys. Rev. Lett. **72**, 1982 (1994)
- [12] C. Ordoñez, L. Ray, and U. van Kolck, Phys. Rev. **C53**, 2086 (1996), hep-ph/9511380
- [13] T. Frederico, V. S. Timoteo, and L. Tomio, Nucl. Phys.



- A653**, 209 (1999), nucl-th/9902052
- [14] D. Eiras and J. Soto, Eur.Phys.J. **A17**, 89 (2003), arXiv:nucl-th/0107009 [nucl-th]
- [15] D. R. Entem and R. Machleidt, Phys. Lett. **B524**, 93 (2002), nucl-th/0108057
- [16] D. R. Entem and R. Machleidt, Phys. Rev. **C66**, 014002 (2002), nucl-th/0202039
- [17] D. R. Entem and R. Machleidt, Phys. Rev. **C68**, 041001 (2003), nucl-th/0304018
- [18] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Eur. Phys. J. **A19**, 125 (2004), nucl-th/0304037
- [19] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Eur. Phys. J. **A19**, 401 (2004), nucl-th/0308010
- [20] E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. **A747**, 362 (2005), arXiv:nucl-th/0405048
- [21] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. **C72**, 054006 (2005), nucl-th/0506005
- [22] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. **C72**, 054002 (2005), nucl-th/0504067
- [23] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. **C74**, 054001 (2006), arXiv:nucl-th/0506047
- [24] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. **C74**, 064004 (2006), arXiv:nucl-th/0507075
- [25] H. Krebs, E. Epelbaum, and U.-G. Meißner, Eur. Phys. J. **A32**, 127 (2007), arXiv:nucl-th/0703087
- [26] R. Higa, M. Pavon Valderrama, and E. Ruiz Arriola, Phys. Rev. **C77**, 034003 (2008), arXiv:0705.4565 [nucl-th]
- [27] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. **C79**, 044001 (2009), arXiv:0809.3186 [nucl-th]
- [28] M. Pavon Valderrama and E. Ruiz Arriola, Phys.Rev. **C83**, 044002 (2011), arXiv:1005.0744 [nucl-th]
- [29] D. R. Entem, E. Ruiz Arriola, M. Pavon Valderrama, and R. Machleidt, Phys. Rev. **C77**, 044006 (2008), arXiv:0709.2770 [nucl-th]
- [30] C. J. Yang, C. Elster, and D. R. Phillips, Phys. Rev. **C77**, 014002 (2008), arXiv:0706.1242 [nucl-th]
- [31] C. J. Yang, C. Elster, and D. R. Phillips, Phys. Rev. **C80**, 034002 (2009), arXiv:0901.2663 [nucl-th]
- [32] C. J. Yang, C. Elster, and D. R. Phillips, Phys. Rev. **C80**, 044002 (2009), arXiv:0905.4943 [nucl-th]
- [33] M. Albaladejo and J. Oller, Phys.Rev. **C84**, 054009 (2011), arXiv:1107.3035 [nucl-th]
- [34] B. Borasoy, E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Eur. Phys. J. **A31**, 105 (2007), arXiv:nucl-th/0611087
- [35] B. Borasoy, E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Eur.Phys.J. **A34**, 185 (2007), arXiv:0708.1780 [nucl-th]
- [36] B. Borasoy, E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Eur. Phys. J. **A35**, 343 (2008), arXiv:0712.2990 [nucl-th]
- [37] B. Borasoy, E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Eur.Phys.J. **A35**, 357 (2008), arXiv:0712.2993 [nucl-th]
- [38] E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Eur.Phys.J. **A40**, 199 (2009), arXiv:0812.3653 [nucl-th]
- [39] E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Eur.Phys.J. **A41**, 125 (2009), arXiv:0903.1666 [nucl-th]
- [40] E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Phys.Rev.Lett. **104**, 142501 (2010), arXiv:0912.4195 [nucl-th]
- [41] E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Eur.Phys.J. **A45**, 335 (2010), arXiv:1003.5697 [nucl-th]
- [42] E. Epelbaum, H. Krebs, D. Lee, and U.-G. Meißner, Phys.Rev.Lett. **106**, 192501 (2011), arXiv:1101.2547 [nucl-th]
- [43] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. **B478**, 629 (1996), nucl-th/9605002
- [44] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. **B424**, 390 (1998), nucl-th/9801034
- [45] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. **B534**, 329 (1998), nucl-th/9802075
- [46] S. R. Beane *et al.*, Phys. Rev. **A64**, 042103 (2001), quant-ph/0010073
- [47] M. C. Birse, Phys. Rev. **C74**, 014003 (2006), arXiv:nucl-th/0507077
- [48] R. Machleidt and D. Entem, J.Phys.G **G37**, 064041 (2010), arXiv:1001.0966 [nucl-th]
- [49] G. P. Lepage(1997), arXiv:nucl-th/9706029
- [50] E. Epelbaum and J. Gegelia, Eur. Phys. J. **A41**, 341 (2009), arXiv:0906.3822 [nucl-th]
- [51] M. Pavon Valderrama, AIP Conf.Proc. **1322**, 205 (2010), arXiv:1009.6100 [nucl-th]
- [52] U. van Kolck, Nucl.Phys. **A645**, 273 (1999), arXiv:nucl-th/9808007 [nucl-th]
- [53] S. R. Beane, P. F. Bedaque, M. J. Savage, and U. van Kolck, Nucl. Phys. **A700**, 377 (2002), nucl-th/0104030
- [54] S. R. Beane, D. B. Kaplan, and A. Vuorinen, Phys. Rev. **C80**, 011001 (2009), arXiv:0812.3938 [nucl-th]
- [55] M. Pavon Valderrama, Phys.Rev. **C83**, 024003 (2011), arXiv:0912.0699 [nucl-th]
- [56] B. Long and U. van Kolck, Annals Phys. **323**, 1304 (2008), arXiv:0707.4325 [quant-ph]
- [57] B. Long and C. Yang, Phys.Rev. **C84**, 057001 (2011), arXiv:1108.0985 [nucl-th]
- [58] M. C. Birse, Phys.Rev. **C76**, 034002 (2007), arXiv:arXiv:0706.0984 [nucl-th]
- [59] M. C. Birse, Eur.Phys.J. **A46**, 231 (2010), arXiv:1007.0540 [nucl-th]
- [60] K. L. Ipson, K. Helmke, and M. C. Birse, Phys.Rev. **C83**, 017001 (2011), arXiv:1009.0686 [nucl-th]
- [61] M. C. Birse, PoS **CD09**, 078 (2009), arXiv:0909.4641 [nucl-th]
- [62] T. R. Hemmert, B. R. Holstein, and J. Kambor, J.Phys.G **G24**, 1831 (1998), arXiv:hep-ph/9712496 [hep-ph]
- [63] E. Epelbaum and U. G. Meißner(2006), arXiv:nucl-th/0609037
- [64] M. Pavon Valderrama and E. Ruiz Arriola, Annals Phys. **323**, 1037 (2008), arXiv:0705.2952 [nucl-th]
- [65] S. R. Beane and M. J. Savage, Nucl.Phys. **A694**, 511 (2001), arXiv:nucl-th/0011067 [nucl-th]
- [66] M. C. Birse, J. A. McGovern, and K. G. Richardson, Phys. Lett. **B464**, 169 (1999), hep-ph/9807302
- [67] T. Barford and M. C. Birse, Phys. Rev. **C67**, 064006 (2003), arXiv:hep-ph/0206146
- [68] J. Gegelia, Phys.Lett. **B429**, 227 (1998)
- [69] S. Fleming, T. Mehen, and I. W. Stewart, Nucl. Phys. **A677**, 313 (2000), nucl-th/9911001
- [70] C. A. Bertulani, H. W. Hammer, and U. Van Kolck, Nucl. Phys. **A712**, 37 (2002), arXiv:nucl-th/0205063
- [71] J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (May 1952)
- [72] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (Jan 1957)
- [73] P. M. Stevenson, Phys. Rev. D **23**, 2916 (Jun 1981)
- [74] M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar, and J. J. de Swart, Phys. Rev. Lett. **82**, 4992 (1999),

- nucl-th/9901054
- [75] P. Büttiker and U.-G. Meißner, Nucl. Phys. **A668**, 97 (2000), hep-ph/9908247
  - [76] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. **C49**, 2950 (1994), nucl-th/9406039
  - [77] V. G. J. Stoks, R. A. M. Kompl, M. C. M. Rentmeester, and J. J. de Swart, Phys. Rev. **C48**, 792 (1993)
  - [78] M. P. Valderrama and E. R. Arriola, Phys. Rev. **C72**, 044007 (2005)
  - [79] N. Kaiser, R. Brockmann, and W. Weise, Nucl. Phys. **A625**, 758 (1997), nucl-th/9706045
  - [80] N. Kaiser, S. Gerstendorfer, and W. Weise, Nucl. Phys. **A637**, 395 (1998), nucl-th/9802071
  - [81] M. C. M. Rentmeester, R. G. E. Timmermans, and J. J. de Swart, Phys. Rev. **C67**, 044001 (2003), nucl-th/0302080
  - [82] D. Shukla, D. R. Phillips, and E. Mortenson, J. Phys. **G35**, 115009 (2008), arXiv:0803.4190 [nucl-th]